The Baum-Connes conjecture, localisation of categories and quantum groups

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Chapter 1

Noncommutative algebraic topology

1.1 What is noncommutative (algebraic) topology?

We can distinguish three stages of noncommutative algebraic topology:

2. Topological invariants of C*-algebras.

In this section we will deal with the second point. A topological invariant for C*-algebras is a functor $F$ on the category of C*-algebras and *-morphisms, with certain formal properties. These properties are

(H) **Homotopy invariance.** If $f_0, f_1 : A \to B$ are two *-morphisms, then a homotopy between them is a *-homomorphism $f : A \to C([0, 1], B)$ such that $ev_t \circ f = f_t$. Homotopy invariance states that if $f_0, f_1$ are homotopic, then $F(f_0) = F(f_1)$.

(E) **Exactness.** For any C*-algebra extension

$$I \hookrightarrow E \to Q$$

the sequence

$$F(I) \to F(E) \to F(Q)$$

is exact.

Since KK-theory does not have this property we also allow functors that are semi-split exact, that is, a sequence (1.2) is exact only for semi-split extensions. We say that the extension (1.1) is semi-split if it has completely positive contractive section $s : Q \to E$. Recall that a map $s : Q \to E$ is positive if and only if $x \geq 0$ implies $s(x) \geq 0$. It is completely positive if and only if $M_n(s) : M_n(Q) \to M_n(E)$ is positive for all $n \geq 0$. A map $s : Q \to E$ is called contractive if $\|s\| \leq 1$.

**Theorem 1.1.** The extension $I \hookrightarrow E \to Q$ with $Q$ nuclear is semi-split.

**Theorem 1.2** (Stinespring). If $s : Q \to E$ is a completely positive contractive map, then there exists a $C^*$-morphism $\pi : Q \to B(\mathcal{H})$, and adjointable contractive isometry $T : E \to \mathcal{H}_E$ ($\mathcal{H}_E$ is a Hilbert $E$-module) such that $s(q) = T^*\pi(q)T$. 
We say that a functor $F$ is split-exact if for every split extension

$$
\begin{array}{ccc}
I & \rightarrow & E \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rightarrow & \rightarrow & \rightarrow \\
Q
\end{array}
$$

(1.3)

The sequence

$$
F(I) \rightarrow F(E) \rightarrow F(Q)
$$

is exact, that is $F(E) \simeq F(I) \oplus F(Q)$.

K-theory is homotopy invariant, exact and split-exact.

**Proposition 1.3.** Let $F$ be a homotopy invariant and (semi-split) exact functor. Then for any (semi-split) extension $I \rightarrow E \rightarrow Q$ there is a natural long exact sequence

$$
\ldots \rightarrow F(S^2Q) \rightarrow F(SI) \rightarrow F(SE) \rightarrow F(SQ) \rightarrow F(I) \rightarrow F(E) \rightarrow F(Q) \quad (1.4)
$$

where $SA := C_0((0,1), A)$ is the suspension functor.

(M) **Morita equivalence** or C*-stability. The third condition for a topological invariant is Morita equivalence. It is of different nature than homotopy invariance and exactness. It is a special feature of the non-commutative world.

For all C*-algebras $A$ the corner embedding

$$
A \rightarrow \mathcal{K}(l^2N) \otimes A
$$

induces an isomorphism $F(A) \simeq F(\mathcal{K} \otimes A)$.

We say that two C*-algebras $A, B$ are Morita equivalent if there exists a two sided Hilbert module $A\mathcal{H}B$ over $A^{op} \otimes B$ such that

$$
(A\mathcal{H}B) \otimes_B (B\mathcal{H}A) \simeq AA \\
(B\mathcal{H}A) \otimes_A (A\mathcal{H}B) \simeq BB
$$

**Theorem 1.4** (Brown–Douglas–Rieffel). Two separable C*-algebras $A, B$ are Morita equivalent, $A \sim_M B$, if and only if $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$.

**Definition 1.5.** A topological invariant for C*-algebras is a functor $F : C*-\text{Alg} \rightarrow \text{Ab}$ which is C*-stable, split exact, semi-split exact and homotopy invariant.

**Theorem 1.6** (Higson). If $F : C*-\text{Alg} \rightarrow \text{Ab}$ is C*-stable and split exact then it is homotopy invariant.

Also if $F : C*-\text{Alg} \rightarrow \text{Ab}$ is semi-split exact and homotopy invariant then it is split exact.

Actually, any topological invariant has many more formal properties like Bott periodicity, Pimsner–Voiculescu exact sequence for crossed product by $\mathbb{Z}$, Connes–Thom isomorphism for crossed products by $\mathbb{R}$, Mayer-Vietoris sequences.

Bott periodicity states that $F(S^2A) \simeq F(A)$ with a specified isomorphism. To prove it one can use two extensions

$$
\mathcal{K} \rightarrow T \rightarrow C(U(1)) \quad (\text{Toeplitz extension})
$$
From the long exact sequence in proposition (1.4) we get boundary maps
\[ F(S^2 A) \to F(K \otimes A) \simeq F(A) \]
The theorem is that this natural map is invertible for any topological invariant.

**Corollary 1.7.** For any topological invariant \( F \), and any split extension
\[ I \hookrightarrow E \twoheadrightarrow Q \]
there is a cyclic six-term exact sequence
\[
\begin{array}{ccc}
F(I) & \to & F(E) \\
& & \to \\
& & F(Q)
\end{array}
\]

If \( F \) is a topological invariant, \( A \) C*-algebra, then \( D \to F(A \otimes D) \) is also a topological invariant. Therefore Bott periodicity is equivalent to the fact, that \( F(C) \simeq F(C_0(\mathbb{R}^2)) \) for all topological invariants \( F \).

### 1.1.1 Kasparov KK-theory
The reason why topological invariants have these nice properties is bivariant K-theory (also called KK-theory or Kasparov theory). Both functors \( B \mapsto KK(A, B) \) and \( A \mapsto KK(A, B) \) are topological invariants.

There is a natural product
\[
KK(A, B) \otimes KK(B, C) \to KK(A, C)
\]
\[
(x, y) \mapsto x \otimes_B y
\]
This turns Kasparov theory into a category \( KK \).

We can characterize \( KK \) using the universal property.

**Definition 1.8.** \( C^* \text{−Alg} \to KK \) is the universal split exact, C*-stable (homotopy) functor.

This means that the functor \( C^* \text{−Alg} \to KK \), which maps a *-homomorphism \( A \to B \) into its class in \( KK(A, B) \), is split exact, and C*-stable. Moreover, for any other functor \( F \) from (separable) C*-algebras to some additive category \( C \) there is a unique factorisation through \( KK \)
\[
\begin{array}{ccc}
C^* \text{−Alg} & \to & KK \\
& F \uparrow & \\
& & C
\end{array}
\]
This abstract point of view explains why KK-theory is so important. It is the universal topological invariant. To be useful, we need existence and a concrete description of KK.

We will describe cycles for \( A, B \). Then homotopies will be cycles in \( KK_0(A, C([0, 1], B)) \).
Next we define \( KK_0(A, B) \) as the set of homotopy classes of cycles. Cycles consist of
• a Hilbert $B$-module $\mathcal{E}$ that is $\mathbb{Z}/2$-graded, $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$
• a $\ast$-homomorphism $\varphi : A \to B(\mathcal{E})^{\text{even}}$
• an adjointable operator $F \in B(\mathcal{E})^{\text{odd}}$

such that
• $F = F^\ast$ (or $(F - F^\ast)\varphi(a) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$)
• $F^2 = 1$ (or $(F^2 - 1)\varphi(a) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$)
• $[F, \varphi(a)] \in \mathcal{K}(\mathcal{E})$ for all $a \in A$.

Addition is the direct sum.

For the odd case we can take

$$\text{KK}_1(A, B) \simeq \text{KK}_0(A, SB) \simeq \text{KK}_0(SA, B)$$

or more concretely drop $\mathbb{Z}/2$-grading in the definition of $\text{KK}_0$.

Kasparov uses Clifford algebras to unify $\text{KK}_0$ and $\text{KK}_1$ and the extend the definition to the real case. We do not treat the real case here but mention the following result

**Theorem 1.9.** Let $A^R$ and $B^R$ be real $C^*$-algebras and let $A^C = A^R \otimes_{\mathbb{R}} \mathbb{C}$, $B^C = B^R \otimes_{\mathbb{R}} \mathbb{C}$ be their complexifications. Then there is a map

$$\text{KK}^R(A^R, B^R) \to \text{KK}^C(A^C, B^C), \quad f^R \mapsto f^C.$$ 

Moreover $f^R$ is invertible if and only if $f^C$ is invertible. In particular $B^R \sim 0$ if and only if $B^C \sim 0$.

### 1.1.2 Connection between abstract and concrete description

Take a cycle $X = (\mathcal{E}, \varphi, F)$ for $\text{KK}_1(A, B)$. Form $E_X = \mathcal{K}(\mathcal{E}) + \varphi(A)(1 + F^2)$. This is a $C^*$-algebra because, modulo $\mathcal{K}(\mathcal{E})$, $P := \frac{1 + F}{2}$ is a projection which commutes with $\varphi(A)$. By construction there is an extension

$$\mathcal{K}(\mathcal{E}) \hookrightarrow E_X \to A'$$

with $\varphi : A \to A'$, $\mathcal{K}(\mathcal{E}) \sim_M I \triangleleft B$. We can assume $\mathcal{E}$ is full and $\varphi(A)$ is injective as a map to $B(\mathcal{E})/\mathcal{K}(\mathcal{E})$. Even $\mathcal{E} = l^2\mathbb{N} \otimes B$ is possible by Kasparov’s Stabilisation Theorem

$$\mathcal{E} \oplus (l^2\mathbb{N} \otimes B) \simeq l^2\mathbb{N} \otimes B$$

After simplifying using $\mathcal{K}(l^2\mathbb{N} \otimes B) \simeq \mathcal{K}(l^2\mathbb{N}) \otimes B$ we get a $C^*$-extension

$$\mathcal{K} \otimes B \hookrightarrow E_X \to A$$

which is semi-split by $a \mapsto P\varphi(a)P$.

Conversely, this process can be inverted using Stinespring’s Theorem, and any semi-split extension

$$\mathcal{K} \otimes B \hookrightarrow E \to A$$

gives a class in $\text{KK}_1(A, B)$.
Thus we can describe $\text{KK}_1(A, B)$ as the set of homotopy classes of semi-split extensions of $A$ by $\mathcal{K} \otimes B$. A deep result of Kasparov replaces homotopy invariance by more rigid equivalence relation: unitary equivalence after adding split extensions. Two extensions are unitarily equivalent if there is a commuting diagram

$$
\begin{array}{ccc}
\mathcal{K} \otimes B & \longrightarrow & E_1 \\
\downarrow \text{Ad}(u) & \approx & \\
\mathcal{K} \otimes B & \longrightarrow & E_2 \\
\end{array}
$$

with $u \in \mathcal{K} \otimes B$ unitary.

**Corollary 1.10.** For any topological invariant $F$ there is a map

$$\text{KK}_1(Q, I) \otimes F_k(Q) \to F_{k+1}(I),$$

where $F_k(A) := F(S^k A)$.

**Proof.** Use the boundary map from proposition (1.3) for the extension associated to a class in $\text{KK}_1(Q, I)$. \qed

Similar construction works in even case. We take

$$\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, \quad \varphi = \varphi^+ \oplus \varphi^-, \quad F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$$

with $u$ unitary.

$$\varphi : A \to B(\mathcal{E}^+), \quad \text{Ad}(u) \circ \varphi^- : A \to B(\mathcal{E}^+)$$

$$\varphi^+(a) - \text{Ad}(u)\varphi^-(a) \in \mathcal{K}(\mathcal{E}^+)$$

for all $a \in A$. From a split extension $\mathcal{K}(\mathcal{E}^+) + \varphi^+(A)$ we get an extension

$$\mathcal{K}(\mathcal{E}^+) \to E \to A$$

that splits by $\varphi^+$ and $\text{Ad}(u) \circ \varphi^-$. Let $F$ be a topological invariant, then

$$F(E) \cong F(B) \oplus F(A),$$

$$F(\varphi^+) - F(\text{Ad}(u) \circ \varphi^-) : F(A) \to F(B) \subset F(E).$$

Hence we get a map

$$\text{KK}_0(A, B) \otimes F(A) \to F(B).$$

Consider two extensions

$$C \to E_2 \to B, \quad B \to E_1 \to A$$

These give a map

$$F(A) \to F(S^{-2}C) \cong F(C).$$

The miracle of the Kasparov product is that this composite map is described by a class in $\text{KK}_0(A, C)$.

**Definition 1.11.** Operator $F$ is **Fredholm** if $\ker(F)$ and $\coker(F)$ have finite dimension.
The operator $F$ in the definition of Kasparov cycles is something like a Fredholm operator. A cycle in $\text{KK}_0(C, C)$ consists of a Hilbert space $H = H_+ \oplus H_-$ and an operator $F: H_+ \to H_-$, $FF^* - \text{id} \in K$, $F^*F - \text{id} \in K$, so $F$ is Fredholm.

The index map gives an isomorphism

$$\text{Index}: \text{KK}_0(C, C) \xrightarrow{\cong} \mathbb{Z}$$

$$\text{Index}(F) = \dim(\ker F) - \dim(\text{coker } F)$$

In the odd case we have $\text{KK}_1(C, C) = 0$.

A pair of *-homomorphisms $f, g: A \to B$ with $(f - g)(A) \subseteq K$ ideal in $B$ gives a morphism $qA \to K$.

$$\text{KK}(A, B) = [qA, B \otimes K] \quad \text{(homotopy classes of *-homomorphisms)}$$

$qA$ is the target of the universal quasi-homomorphism.

1.1.3 Relation with K-theory

KK-theory is very close to K-theory. If some construction gives a map $K_*(A) \to K_*(B)$ it probably gives a class in $\text{KK}_*(A, B)$.

**Theorem 1.12.** $\text{KK}_*(C, A) \simeq K_*(A)$.

The proof requires the concrete description of KK.

Hence there is a canonical map

$$\gamma: \text{KK}_*(A, B) \to \text{Hom}(K_*(A), K_*(B)).$$

In many cases, this map is injective and has kernel $\text{Ext}^1(K_*(A), K_{*+1}(B))$.

Take $\alpha \in \text{KK}_1(Q, I)$, $\alpha = [I \to E \to Q]$. Assume $\gamma(\alpha) = 0$. There is an exact sequence

$$
\begin{array}{c}
K_0(I) \
\xrightarrow{\gamma(\alpha)} \
K_0(E) \xrightarrow{} K_0(Q) \\
\end{array}
\begin{array}{c}
K_1(Q) \xleftarrow{} K_1(E) \xrightarrow{} K_1(I)
\end{array}
$$

We get an extension of $\mathbb{Z}/2$-graded abelian groups.

$$K_*(I) \to K_*(E) \to K_*(Q).$$

This defines a natural map

$$\text{KK}_*(A, B) \supset \ker \gamma \to \text{Ext}^1(K_{*+1}(A), K_*(B)).$$

In many cases this map and $\gamma$ provide the Universal Coefficient Sequence (1.5)
Theorem 1.13. Let $\mathcal{B}$ be the smallest category of separable C*-algebras closed under suspensions, semi-split extensions, KK-equivalence, tensor products, and containing $\mathbb{C}$. Then there exists a natural exact sequence

$$\text{Ext}^1(K_{*+1}A, K_*B) \rightarrow KK_*(A, B) \rightarrow \text{Hom}(K_*, A, K_*B)$$

(1.5)

for $A, B \in \mathcal{B}$

**Corollary 1.14.** Let $X$ and $Y$ be locally compact spaces. If $K^*(X \setminus \{x\}) \simeq K^*(Y \setminus \{y\})$ then $F(C_0(X \setminus \{x\})) \simeq F(C_0(Y \setminus \{y\}))$ for any topological invariant for C*-algebras.

**Proof.** Denote $\tilde{X} := X \setminus \{x\}$, $\tilde{Y} := Y \setminus \{y\}$.

$$\alpha: K^*(X \setminus \{x\}) \simeq K^*(C_0(X \setminus \{x\})) \xrightarrow{\sim} K^*(C_0(Y \setminus \{y\}))$$

By the universal coefficients theorem, $\alpha$ lifts to $\hat{\alpha} \in KK_0(C_0(\tilde{X}), C_0(\tilde{Y}))$. Because $\text{Ext}^1 \circ \text{Ext}^1 = 0$ we know that $\hat{\alpha}$ is invertible. Since KK is universal, $F(\hat{\alpha})$ is invertible for any topological invariant $F$.

There are analogies and contrasts between homotopy theory and noncommutative topology. We will summarize them in a table:

<table>
<thead>
<tr>
<th>Homotopy theory</th>
<th>Noncommutative topology</th>
</tr>
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<tbody>
<tr>
<td>Spaces</td>
<td>C*-algebras</td>
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<tr>
<td>Stable homotopy category</td>
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</tr>
<tr>
<td>Stable homotopy groups of spheres</td>
<td>Morphisms from $C$ to $C$ in $KK$</td>
</tr>
<tr>
<td>$\pi_<em>(S^0) = \text{Mor}_</em>(\text{pt}, \text{pt})$</td>
<td>$KK^*(C, C) = \mathbb{Z}[\beta, \beta^{-1}], \deg(\beta) = 2$</td>
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<tr>
<td>Homology $H_*(-)$</td>
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<td>Adams spectral sequence</td>
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<td>Always works but complicated</td>
<td>Universal coefficients theorem for KK</td>
</tr>
<tr>
<td>Interesting topology - no analysis</td>
<td>Simple topology - interesting analysis</td>
</tr>
</tbody>
</table>

1.2 Equivariant theory

In equivariant bivariant Kasparov theory additional symmetries create interesting topology, making tools from homotopy theory more relevant.

What equivariant situations are being considered?

- Group actions (of locally compact groups)
- Bundles of C*-algebras $(A_x)_{x \in X}$ over some space $X$
- Locally compact groupoids
- Locally compact quantum group actions (Baaj-Skandalis)
- C*-algebras over non-Hausdorff space (Kirchberg)

In each case, there is an equivariant K-theory with similar properties as the nonequivariant one, with a similar concrete description – add equivariance condition – and an universal property.
Proposition 1.15. If \( G \) is a group, then \( KK^G(\mathbb{C}, \mathbb{C}) \) is a graded commutative ring, and the exterior product coincides with composition product. Furthermore \( KK^G(\mathbb{C}, \mathbb{C}) \) acts on \( KK^G(A, B) \) for all \( A, B \in \mathbb{C}^* - \text{alg}_G \) by exterior product.

Let \( \mathcal{G} \) be a groupoid, and \( A \) a C*-algebra. Then we say that \( \mathcal{G} \) acts on \( A \), \( \mathcal{G} \curvearrowright A \), if \( A \) is a bundle over \( \mathcal{G}^0 \), \( \mathcal{G} \) acts fiberwise on this bundle. Continuity of the action is expressed by the existence of a bundle isomorphism \( \alpha : s^*A \to r^*A \), where \( r, s \) are the range and source maps of \( \mathcal{G} \).

\[ \mathcal{G}^1 \xrightarrow{r} \mathcal{G}^0, \quad s^*A \xrightarrow{\alpha} r^*A, \quad (s^*A)_y = A_x. \]

Let \( \mathcal{G} \) be a groupoid, and \( A \) a C*-algebra. Then we say that \( \mathcal{G} \) acts on \( A \), \( \mathcal{G} \curvearrowright A \), if \( A \) is a bundle over \( \mathcal{G}^0 \), \( \mathcal{G} \) acts fiberwise on this bundle. Continuity of the action is expressed by the existence of a bundle isomorphism \( \alpha : s^*A \to r^*A \), where \( r, s \) are the range and source maps of \( \mathcal{G} \).

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\[ \mathcal{G}^1 \xrightarrow{r} \mathcal{G}^0, \quad s^*A \xrightarrow{\alpha} r^*A, \quad (s^*A)_y = A_x. \]

We fix some category of C*-algebras with symmetries, equivariant *-homomorphisms. We denote it \( \mathbb{C}^* - \text{alg}_G \). We study functors \( F \) from \( \mathbb{C}^* - \text{alg}_G \) to an additive category, such that if \( I \to E \to Q \) is a split extension in \( \mathbb{C}^* - \text{alg}_G \), then \( F(I) \to F(E) \to F(Q) \) is a split extension in \( \mathbb{C}^* - \text{alg}_G \), then

\[ F(\mathcal{G}^1) \xrightarrow{r} \mathcal{G}^0, \quad s^*A \xrightarrow{\alpha} r^*A, \quad (s^*A)_y = A_x. \]

Split exactness is considered for equivariant *-homomorphisms in extensions, and the section is supposed to be also equivariant.

Let \( A \) be an object of \( \mathbb{C}^* - \text{alg}_G \) and \( H \) a \( G \)-equivariant full Hilbert module over \( A \). Then \( F \) is stable if both maps

\[ A \to \mathcal{K}(H \oplus A) \to \mathcal{K}(H) \]

coming from inclusions of Hilbert modules \( A \hookrightarrow H \oplus A \hookrightarrow H \) become isomorphisms after applying \( F \)

\[ F(A) \to F(\mathcal{K}(H \otimes A)) \to F(\mathcal{K}(H)) \]

In the cases mentioned above, \( KK^G \) is the universal split-exact stable functor on \( \mathbb{C}^* - \text{alg}_G \) (separable), that is, any other functor with this properties factors uniquely through \( KK^G \).

\[ KK^G(A, B) \times F(A) \to F(B) \]

1.2.1 Tensor products

The following discussion also shows how the universal property of KK can be used to construct functors between KK-categories and to prove adjointness relations between such functors.

The minimal tensor product of two \( G \)-C*-algebras is again a \( G \)-C*-algebra if \( G \) is a groupoid. Here we use the diagonal action of the groupoid. This yields a functor

\[ \otimes : \mathbb{C}^* - \text{alg}_G \times \mathbb{C}^* - \text{alg}_G \to \mathbb{C}^* - \text{alg}_G, \quad (A, B) \mapsto A \otimes B. \]

For a group(oid) diagonal action of \( G \) on \( A \otimes B \), if \( G \) acts on \( A, B \). This descends to

\[ \mathbb{C}^* - \text{alg} \times \mathbb{C}^* - \text{alg}_G \to \mathbb{C}^* - \text{alg}_G \]

\[ KK \times KK^G \to KK^G \]
We will provide the concrete description. Let $\beta \in \text{KK}^G(B_1, B_2)$, $\alpha \in \text{KK}^G(A_1, A_2)$. The tensor product is given by

$$\alpha \otimes \beta = (\alpha \otimes \text{id}_{B_2}) \circ (\text{id}_{A_1} \otimes \beta) = (\text{id}_{A_2} \otimes \beta) \circ (\alpha \otimes \text{id}_{B_1}).$$

![Diagram of tensor product](image)

In the abstract approach we fix $A$ and consider functor

$$\text{C}^* - \text{alg}_G \rightarrow \text{C}^* - \text{alg}_G \rightarrow \text{KK}^G$$

$B \mapsto A \otimes B \mapsto A \otimes B$ which is split-exact, stable. The functor $\text{KK}^G \rightarrow \text{KK}^G$ exists by the universal property.

In general, if $F_1, F_2: \text{C}^* - \text{alg}_G \rightarrow \text{Ab}$ are split exact and stable, and $\Phi: F_1 \rightarrow F_2$ is a natural transformation, then there exist $\overline{F}_1, \overline{F}_2: \text{KK}^G \rightarrow \text{Ab}$ and a natural transformation $\overline{\Phi}: \overline{F}_1 \rightarrow \overline{F}_2$ such that the following diagram commutes for $\alpha \in \text{KK}^G(A_1, A_2)$

![Diagram of natural transformation](image)

The diagram above commutes for $\alpha, \beta$ KK-morphisms provided it commutes for $\alpha, \beta$ equivariant *-homomorphisms. This is a part of the universal property of $\text{KK}^G$.

If $A, B$ are $G$-C*-algebras, then $A \otimes B$ gives a tensor product in $\text{KK}^G$. Descent functor $\text{KK}^G \rightarrow \text{KK}$ is obtained by taking crossed products on objects and *-homomorphisms.

The functor

$$A \mapsto G \rtimes_r A$$

is split-exact, stable, so it descends to $\text{KK}^G$

$$\text{KK}^G(A, B) \rightarrow \text{KK}(G \rtimes_r A, G \rtimes_r B).$$

If $H \leq G$ is a closed subgroup, $H \ltimes A$, then $\text{Ind}_{H}^{G} A \ltimes G$, where

$$\text{Ind}_{H}^{G} A := \{f \in C_0(G, A) \mid f(gh) = (\alpha_h f)(g), \|f\| \in C_0(G/H)\}.$$

(On the level of spaces the induction is $\text{Ind}_{H}^{G}: X \mapsto G \times_{H} X$). It induces

$$\text{Ind}_{H}^{G}: \text{KK}^{H} \rightarrow \text{KK}^{G}.$$

The composition

$$\text{C}^* - \text{alg}_{H} \xrightarrow{\text{Ind}_{H}^{G}} \text{C}^* - \text{alg}_{G} \xrightarrow{-\kappa_{r} G} \text{C}^* - \text{alg}_{G} \xrightarrow{-\kappa_{r} H} \text{C}^* - \text{alg}\quad A \xrightarrow{\text{Ind}_{H}^{G}} \text{Ind}_{H}^{G} A \xrightarrow{-\kappa_{r} G} G \rtimes_r \text{Ind}_{H}^{G} A \xrightarrow{-\kappa_{r} H} H \rtimes_r A$$

![Diagram of composition](image)
becomes a natural isomorphism in $\text{KK}(H \times_I A, G \times_I \text{Ind}_H^G A)$ for $H$-equivariant *-homomorphisms or for $\text{KK}^H$-morphisms (equivalent by the universal property of $\text{KK}^H$).

For open $H \leq G$

$$\text{KK}^G(\text{Ind}_H^G A, B) \simeq \text{KK}^H(A, \text{Res}_H^G B)$$

the following compositions

$$\text{Ind}_H^G \text{Res}_H^G A \simeq C_0(G/H) \otimes A \hookrightarrow K((l^2(G/H))) \otimes A \sim_{M.E.} A.$$

$$B \hookrightarrow \text{Res}_H^G \text{Ind}_H^G B$$

are natural for *-homomorphisms, hence KK-morphisms.

### 1.3 KK as triangulated category

The category KK is additive, but not abelian. However it can be triangulated. This notion is motivated by examples in homological algebra: derived category of an abelian category, homotopy category of chain complexes over an additive category, homotopy category of spaces.

The additional structure in a triangulated category consists of

- **translation/suspension functor.** In $\text{KK}^G$:

  $$A[-n] := C_0(\mathbb{R}^n) \otimes A, \quad \text{for } n \geq 0.$$

- **exact triangles**

  $$A \to B \to C \to A[1].$$

Merely knowing the KK-theory class of $i, p$ in a C*-algebra extension

$$I \overset{i}{\longrightarrow} E \overset{p}{\longrightarrow} Q$$

does not determine the boundary maps. This requires a class in $\text{KK}_1(Q, I)$.

**Definition 1.16.** A diagram

$$A \stackrel{u}{\longrightarrow} B \stackrel{v}{\longrightarrow} C \stackrel{w}{\longrightarrow} A[1]$$

in $\text{KK}^G$ is called an exact triangle if there are KK-equivalences $\alpha, \beta, \gamma$ such that the following diagram commutes

$$\begin{array}{ccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\
\alpha & \simeq & \beta & \simeq & \gamma & \simeq & \alpha[1] & \simeq \\
A' & \xrightarrow{[i]} & B' & \xrightarrow{[p]} & C' & \xrightarrow{[\delta]} & A'[1]
\end{array}$$

where $A' \to B' \to C'$ is a C*-algebra semi-split extension, and $\delta$ is its class in $\text{KK}_1(C, A)$.

**Proposition 1.17.** With this additional structure $\text{KK}^G$ is a triangulated category.

In general the structure of a triangulated category consists of an additive category $T$, an automorphism $\Sigma: T \to T$, and a class $\mathcal{E} \subseteq \text{Triangles}(T)$ of exact triangles.
Example 1.18. Homotopy category of chain complexes over $A$

$$\Sigma(C_n, d_n) = (C_{n-1}, -d_{n-1}), \quad \Sigma(f_n) = f_{n-1}(f_n^{-1} \text{ chain map})$$

Triangle is exact if it is isomorphic to an exact triangle

$$I \rightarrow E \rightarrow Q,$$

where $I, E, Q$ are chain complexes, $i, p$ are chain maps, $s$ is a morphism in $A$. Define

$$\delta_s : Q \rightarrow I[1], \quad \delta_s = d^E \circ s - s \circ d^Q$$

Then

$$I \rightarrow E \rightarrow Q \xrightarrow{\delta_s} I[1]$$

is an extension triangle. However the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{s} & Q \\
\downarrow{d^E} & & \downarrow{d^Q} \\
E[1] & \xrightarrow{s[1]} & Q[1]
\end{array}
$$

is not commutative.

It is easier to work with mapping cone triangles instead of extension triangles. Let $f : A \rightarrow B$ be a *-homomorphism. Then we define its cone as the algebra

$$C_f := \{(a, b) \in A \oplus C_0([0, 1]) \otimes B \mid f(a) = b(1)\}$$

$$SB \rightarrow C_f \rightarrow A$$

is a C*-algebra semi-split extension.

On the level of spaces, if $f : X \rightarrow Y$ is a map, then

$$C_f = x \times [0, 1] \amalg Y/(x, 0) \sim (x', 0) \sim (s, t), \ (x, 1) \sim f(x)$$

$K_*(C_f)$ gives a relative K-theory for $f$. The Puppe exact sequence for $F$ is a long exact sequence

$$\ldots \rightarrow F(SC_f) \rightarrow F(SA) \rightarrow F(SB) \rightarrow F(C_f) \rightarrow F(A) \xrightarrow{F(f)} F(B)$$

Long exact sequence, say for KK, are often established by first checking exactness of the Puppe sequence, then getting other extensions from that.

Definition 1.19. A **mapping cone triangle** is a triangle that is isomorphic to

$$SB \rightarrow C_f \rightarrow A \xrightarrow{f} B$$

for some $f$ in $\text{KK}^G$.

Theorem 1.20. A triangle in $\text{KK}^G$ is exact (isomorphic to an exact triangle) if and only if it is isomorphic to a mapping cone triangle.
Proof. Consider extension

\[
\begin{array}{ccc}
SQ & \xrightarrow{\delta} & I \\
\downarrow & & \downarrow i \\
SQ & \xrightarrow{c} & E \xrightarrow{p} Q
\end{array}
\]

Exact sequences for KK are established by showing that \( I \hookrightarrow C_p \) is a KK-equivalence if the extension is semi-split.

Cuntz-Skandalis: exact triangles are isomorphic to mapping cone triangles. Conversely, consider a mapping cylinder for a \(*\)-homomorphims \( f: A \to B \), that is

\[
Z_f := A \oplus_B B \otimes C([0,1]),
\]

and two extensions

\[
\begin{array}{ccc}
SB & \xrightarrow{\delta} & Cf \\
\downarrow & & \downarrow i \\
SB & \xrightarrow{c} & A \xrightarrow{j} B
\end{array}
\]

where \( j: A \to Z_f \) is a homotopy equivalence. If the triangle

\[
C[-1] \to A \to B \to C
\]

is exact, then it is isomorphic to

\[
SY \to Cf \to X \xrightarrow{f} Y.
\]

Next we get an extension triangle

\[
SX \xrightarrow{-Sf} SY \to Cf \to X,
\]

so the triangle

\[
B[-1] \xrightarrow{-w} C[-1] \xrightarrow{u} A \xrightarrow{v} B
\]

is exact.

\[
\square
\]

1.4 Axioms of a triangulated categories

Triangulated category consists of an additive category with suspension automorphism and a class of exact triangles. These are supposed to satisfy the following axioms (TR0-TR4)

(TR0) If a triangle is isomorphic to an exact triangle, then it is exact. Triangles of the form

\[
0 \to A \xrightarrow{id} A \to 0
\]

are exact.

(TR1) Any morphism \( f: A \to B \) can be embedded in an exact triangle

\[
\Sigma B \to C \to A \xrightarrow{f} B
\]

(we will see that this triangle is unique up to isomorphism and call \( C \) a cone for \( f \)).
The best proof of this for KK uses extension triangles. Let \( f \in KK_0(A, B) \simeq KK_1(\Sigma A, B) \simeq \text{Ext}(\Sigma A, B) \). Hence \( f \) generates a semi-split extension

\[
\begin{align*}
B \otimes K & \to E \to \mathcal{G}A, \\
K(\mathcal{H}_B) & 
\end{align*}
\]

which yields an extension triangle

\[
\begin{array}{c}
\Sigma^2 A \\
\approx \text{Bott} \\
A \\
\end{array} \to \begin{array}{c}
K(\mathcal{H}_B) \\
\approx \text{M.E.} \\
B
\end{array} \to E \to \Sigma A
\]

Now rotate this sequence to bring \( f \) to the right place.

(TR2) The triangle

\[
\begin{array}{c}
\Sigma B \\
\to C \\
A \\
\end{array} \to \begin{array}{c}
w \\
u \\
v
\end{array} \to B
\]

is exact if and only if the triangle

\[
\begin{array}{c}
\Sigma A \\
\to \Sigma B \\
A
\end{array} \to \begin{array}{c}
w \\
u \\
u
\end{array} \to C \to A
\]

is exact. We can get rid of some minus signs by taking

\[
\begin{array}{c}
\Sigma A \\
\to \Sigma B \\
A
\end{array} \to \begin{array}{c}
w \\
u \\
u
\end{array} \to C \to A
\]

By applying three times we get that

\[
\Sigma^2 B \to \Sigma C \to \Sigma A \to \Sigma^2 B
\]

is exact. The reason for a sign is that the suspension of a mapping cone triangle for \( f \) is the mapping cone triangle for \( \Sigma f \) but this involves a coordinate flip on \( \mathbb{R}^2 \) on \( \Sigma^2 B = C_0(\mathbb{R}^2, B) \), which generates a sign.

**Definition 1.21.** A functor \( F \) from a triangulated category to an abelian category is called homological if

\[
F(\Sigma C) \to F(A) \to F(B)
\]

is exact for any exact triangle

\[
\Sigma B \to C \to A \to B.
\]

**Example 1.22.** If \( F \) is a semi-split exact, split exact, \( C^* \)-stable functor on \( C^* - \text{alg} \), then its extension to \( KK \) is homological.

**Proposition 1.23.** If \( F \) is homological, then any exact triangle yields a long exact sequence

\[
\ldots F_n(C) \to F_n(A) \to F_n(B) \to F_{n-1}(C) \to \ldots
\]

where \( F_n(A) := F(\Sigma^n A) \), \( n \in \mathbb{Z} \).
Proof. Use axiom (TR2).

(TR3) Consider a commuting diagram with exact rows

\[
\begin{array}{ccc}
\Sigma B & \longrightarrow & C \\
\downarrow \Sigma \beta & & \downarrow \alpha \\
\Sigma B' & \longrightarrow & C'
\end{array}
\]

There exists \( \gamma : C \to C' \) making the diagram commutative (but it is not unique).

We will prove (TR3) for KK. We may assume that rows are mapping cone triangles

\[
\begin{array}{ccc}
\Sigma B & \longrightarrow & C_f \\
\downarrow \Sigma \beta & & \downarrow \alpha \\
\Sigma B' & \longrightarrow & C'_f
\end{array}
\]

We know that \( \alpha \) is a KK-cycle for \( A \to A' \), \( \beta \) is a KK-cycle for \( B \to B' \), and there exists a homotopy \( H \) from \( \beta \circ f \) to \( f' \circ \alpha \) (because the classes \([\beta \circ f] = [f' \circ \alpha] \) in KK).

Denote

\[
\begin{aligned}
\alpha &= (H^\alpha_B, \varphi^\alpha, F^\alpha \in B(\mathcal{H}^\alpha)), \\
\beta &= (H^\beta_B, \varphi^\beta, F^\beta \in B(\mathcal{H}^\beta)), \\
H &= (H^H_C([0,1],B'), \varphi^H, F^H \in B(\mathcal{H}^H)),
\end{aligned}
\]

such that

\[
\begin{aligned}
H|_0 &= \beta \circ f = (H^\beta, \varphi^\beta \circ f, F^\beta), \\
H|_1 &= f' \circ \alpha = (H^\alpha \otimes f', B', \varphi^\alpha \otimes id_{B'}, F^\alpha \otimes id_{B'}). \end{aligned}
\]

Then

\[
\mathcal{H}^\beta \otimes C([0,\frac{1}{2}]) \oplus_{\mathcal{H}^\beta} at \frac{1}{2} \mathcal{H}^H \oplus_{\mathcal{H}^\alpha \otimes f'} B' \mathcal{H}^\alpha
\]

is a mapping cone of \( f' \). Now \( \varphi^\beta \otimes C([0,\frac{1}{2}]), \varphi^H, \varphi^\alpha \) glue to \( \varphi^\gamma : A \to B(\mathcal{H}) \). Similarly for \( F \).

Many results use only axioms (TR0)-(TR3). The last one, (TR4) will be given at the end. Before that we will prove

**Proposition 1.24.** Let \( D \) be an object of a category \( T \). Then the functor \( A \to T(D, A) \) is homological. Dually \( A \to T(A, B) \) is cohomological for every object \( B \) in \( T \).

**Proof.** Let

\[
\Sigma B \to C \to A \to B
\]

be an exact triangle in \( T \). We have to verify the exactness of

\[
T(D, C) \to T(D, A) \to T(D, B).
\]

We use the fact that in an exact triangle, the composition \( C \to A \to B \) is zero. Hence

\[
\begin{array}{ccc}
T(D, C) & \longrightarrow & T(D, A) \\
\downarrow & & \downarrow \\
T(D, C) & \longrightarrow & T(D, B)
\end{array}
\]

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Now we use (TR3) to complete diagram

\[
\begin{array}{ccc}
0 & \rightarrow & D \\
\downarrow & & \downarrow f \\
\Sigma B & \rightarrow & C \\
\downarrow & & \downarrow \\
& A & \rightarrow B
\end{array}
\]

with \( \hat{f} : D \rightarrow C \).

Example 1.25. \( \text{KK}^G(\mathcal{A}, D) \) is homological, and \( \text{KK}^G(D, \mathcal{A}) \) is cohomological.

Lemma 1.26 (Five lemma). Consider morphism of exact triangles

\[
\begin{array}{ccc}
\Sigma B & \rightarrow & C \\
\downarrow & & \downarrow \gamma \\
& \alpha & \rightarrow \beta \\
\downarrow & & \downarrow \\
& A & \rightarrow B
\end{array}
\]

If two of \( \alpha, \beta, \gamma \) are invertible, then so is the third.

Proof. Assume \( \alpha, \beta \) are invertible. Then \( T(D, \alpha), T(D, \beta), \) and \( T(D, \Sigma \alpha), T(D, \Sigma \beta) \) are invertible. We can use exact sequences from the proposition (1.24) and write a diagram

\[
\begin{array}{cccccc}
T(D, \Sigma A) & \rightarrow & T(D, \Sigma \beta) & \rightarrow & T(D, C) & \rightarrow & T(D, A) & \rightarrow & T(D, B) \\
\downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong \\
T(D, \Sigma \alpha) & \rightarrow & T(D, \beta) & \rightarrow & T(D, \gamma) & \rightarrow & T(D, \alpha) & \rightarrow & T(D, \beta) \\
\downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong \\
T(D, G \mathcal{A}') & \rightarrow & T(D, \Sigma \mathcal{B}') & \rightarrow & T(D, \mathcal{C}') & \rightarrow & T(D, \mathcal{A}') & \rightarrow & T(D, \mathcal{B}')
\end{array}
\]

Rows are exact chain complexes, so the five lemma yields \( T(D, \gamma) \) invertible.

Proposition 1.27. Let \( f : A \rightarrow B \) be a morphism. There is up to isomorphism a unique exact triangle

\[
\Sigma B \rightarrow C \rightarrow A \xrightarrow{f} B
\]

Proof. Existence comes from (TR1). From the (TR3) we get \( \gamma \) in the following diagram

\[
\begin{array}{ccc}
\Sigma B & \rightarrow & C \\
\downarrow & & \downarrow \gamma \\
& A & \rightarrow B \\
\downarrow & & \downarrow \\
\Sigma B & \rightarrow & C' \\
& A & \rightarrow B
\end{array}
\]

From the five lemma (1.26) we get that \( \gamma \) is invertible, which gives uniqueness.

Lemma 1.28. Let

\[
\Sigma B \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B
\]

be an exact triangle. Then

1. \( B = 0 \) if and only if \( v \) is invertible
2. \( u = 0 \) if and only if \( C \rightarrow A \rightarrow B \) is a split extension \( (A \simeq C \oplus B) \)
Proof. 1. If \( v \) is invertible, then
\[
0 \to C \xrightarrow{v} A \to 0
\]
is an exact triangle by (TR0) and
\[
\begin{array}{cccc}
0 & \rightarrow & C & \rightarrow & A & \rightarrow & 0 \\
\| & \| & \searrow & \| & \| & \searrow & \\
0 & \rightarrow & A & \rightarrow & A & \rightarrow & 0
\end{array}
\]
For the converse we use long exact sequence for \( T(D, -) \). We have \( T(D, B) = 0 \) if and only if \( T(D, v) \) invertible. Then we use the Yoneda lemma.

2. If \( A \to B \) is split epimorphism, then \( B \to \Sigma^{-1}C \) vanishes because \( A \to B \to \Sigma^{-1}C \) vanishes.
Assume \( u = 0 \). We use exactness of
\[
T(B, A) \to T(B, B) \to T(B, \Sigma^{-1}C)
\]
to get \( s: B \to A \)
\[
s \mapsto \text{id}_B \mapsto 0
\]
which gives a section for \( w: A \to B, w \circ s = \text{id}_B \).
Exactness of
\[
\ldots \to T(D, C) \to T(D, A) \to T(D, B) \to \ldots
\]
implies that \( T(D, v) \) and \( T(D, s) \) give isomorphism
\[
T(D, C) \oplus T(D, B) \to T(D, A)
\]
for all \( D \), so \( (s, v) \) give isomorphism \( C \oplus B \xrightarrow{\sim} A \). Given \( B, C \) embed \( B \oplus C \to B \) in an exact triangle
\[
\Sigma B \to D \to B \oplus C \to B
\]
Since \( B \oplus C \xrightarrow{w} B \) is an epimorphism we have \( u = 0 \). From the long exact sequence
\[
\ldots \to T(X, D) \to T(X, B \oplus C) \to T(X, B) \to \ldots
\]
we get \( T(X, D) \simeq T(X, C) \) for all \( X \in T \), so \( D \simeq C \).

\[
\square
\]

**Proposition 1.29.** If
\[
\Sigma B_i \to C_i \to A_i \to B_i
\]
are exact triangles for all \( i \in I \), and direct sums exist, then
\[
\bigoplus_{i \in I} \Sigma B_i \to \bigoplus_{i \in I} C_i \to \bigoplus_{i \in I} A_i \to \bigoplus_{i \in I} B_i
\]
is exact. The same holds for products.

**Definition 1.30.** A square
\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\beta \downarrow & & \downarrow \beta' \\
X' & \xrightarrow{\alpha'} & Y'
\end{array}
\]
is called homotopy Cartesian with differential $\gamma: \Sigma Y' \to X$ if

$$\Sigma Y' \xrightarrow{\gamma} X \xrightarrow{\alpha} Y \oplus X' \xrightarrow{\beta'} Y'$$

is exact.

Given $\alpha, \beta$ in the definition we get $\alpha', \beta', \gamma'$ unique up to isomorphism by embedding $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in an exact triangle (homotopy pushout). Dually, given $\alpha', \beta'$ there are $\alpha, \beta, \gamma$ unique up to isomorphism (homotopy pullback).

**Definition 1.31.** Let $(A_n, \alpha_n^{n+1}: A_n \to A_{n+1})_{n \in \mathbb{N}}$ be an inductive system in a triangulated category. We define its homotopy colimit $\operatorname{holim} \to (A_n, \alpha_n^{n+1}: A_n \to A_{n+1})_{n \in \mathbb{N}}$ as the desuspended cone of the map

$$\bigoplus_{n \in \mathbb{N}} A_n \xrightarrow{id - S} \bigoplus_{n \in \mathbb{N}} A_n \xrightarrow{\alpha_n^{n+1}} \bigoplus_{n \in \mathbb{N}} \Sigma^{-1} A_n$$

It is unique up to isomorphism but not functorial.

**Proposition 1.32.** Let $F: T \to \text{Ab}$ be homological and commuting with $\oplus$, then

$$F(\operatorname{holim} A_n) = \lim_{\to} F(A_n)$$

If $\bar{F}: T \to \text{Ab}^\text{op}$ is contravariant cohomological and $\bar{F}(\bigoplus A_n) = \prod \bar{F}(A_n)$, then there is an exact sequence

$$\lim^1 \bar{F}(A_n) \to \bar{F}((\operatorname{holim} A_n)) \to \lim \bar{F}(A_n)$$

**Proof.** Apply $F$ to the exact triangle defining $\operatorname{holim}$

$$\bigoplus F_n(A_m) \xrightarrow{id - S} \bigoplus F_n(A_m) \to F_n(\operatorname{holim} A_n) \to \bigoplus F_{n-1}(A_m) \to \bigoplus F_{n-1}(A_m) \to \ldots$$

$$\ker(id - S) = \lim_{\to} F_n(A_m), \quad \ker(id - S) = 0.$$

**Fact 1.33.** If $A \to B \to C \to D \to E$ is exact, then

$$\ker(A \to B) \to C \to \ker(D \to E)$$

is an extension.

**Example 1.34.** Let $e: A \to A$ be an idempotent morphism. Then $\operatorname{holim}(A, e: A \to A)$, $A \xrightarrow{e} A \xrightarrow{e} \ldots$ is a range object for $e$ and $A \simeq eA \oplus (1 - e)A$.

There are two questions concerning $C^*$-algebras:
1. Let

\[
\begin{array}{c}
X \xrightarrow{\alpha} Y \\
\downarrow \beta \quad \downarrow \gamma' \\
X' \xrightarrow{\alpha'} Y'
\end{array}
\]

be a pullback diagram of C*-algebras, so that

\[X = \{(x', y) \in X' \times Y | \alpha'(x') = \beta'(y)\} \]

When is this image in KK homotopy Cartesian?

2. Let \((A_n, \alpha_n)\) be an inductive system of C*-algebras. Is \(\lim\rightarrow (A_n, \alpha_n)\) also a homotopy colimit?

Ad 1. Compare \(X\) to the homotopy pullback

\[H = \{(x', y', y) \in X' \times C(I, Y) \times Y | \alpha'(x') = y'(0), \beta'(y) = y'(1)\}\]

\(H\) is a part of an extension

\[\Sigma Y' \xrightarrow{\alpha'} H \xrightarrow{\beta'} X' \oplus Y\]

which is semisplit. Its class in \(\text{KK}_1(X' \oplus Y, \Sigma Y) \simeq \text{KK}_0(X' \oplus Y, Y')\) is \((\beta', -\alpha')\), so \(H\) is a homotopy pullback.

**Definition 1.35.** The pullback square is admissible if \(X \to H\) is a KK-equivalence.

**Proposition 1.36.** If \(\alpha'\) is a semisplit epimorphism then so is \(\alpha\), and the pullback square is admissible. Thus we get a long exact sequence

\[\ldots \to F_n(X) \to F_n(X') \oplus F_n(Y) \to F_n(Y') \to \ldots\]

for any semisplit-exact C*-stable homotopy functor.

**Proof.** If \(\alpha'\) is semisplit epimorphism, then \(\alpha\) is a semisplit epimorphism.
The map \( X' \to Z_{\alpha'} \) is a homotopy equivalence, and \( K \to C_{\alpha'} \) is a KK-equivalence because the extension \( K \to X' \to Y' \) is semisplit. Now use five lemma in KK to get that \( X \to H \) is a KK-equivalence.

Ad 2. If all \( A_n \) are nuclear, then \( \lim\to(A_n, \alpha_n) \) is a homotopy colimit.

There is a fourth axiom of triangulated categories which is about exactness properties of cones of maps.

\[(TR4)\]

Given solid arrows so that \((\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3), (\gamma_1, \gamma_2, \gamma_3)\) are exact triangles, we can find exact triangle \((\delta_1, \delta_2, \delta_3)\) making the diagram commute.

We should warn the reader that the arrows are reversed here compared to the previous convention.

There are equivalent versions of the axiom \((TR4)\):

\[(TR4')\] Every pair of maps

\[X \xrightarrow{\alpha} Y\]

\[X'\]

can be completed to a morphism of exact triangles

\[X \xrightarrow{\alpha} Y \to Z \to \Sigma X\]

\[X' \xrightarrow{\alpha'} Y' \to Z \to \Sigma X'\]

such that the first square is homotopy Cartesian with differential \( Y' \to \Sigma X' \).

\[(TR4'')\] Given a homotopy Cartesian square

\[X \xrightarrow{} Y\]

\[X' \xrightarrow{} Y'\]
and differential $\delta : Y' \to \Sigma X$, it can be completed to a morphism of exact triangles

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
X' & \rightarrow & Y'
\end{array}
\quad \begin{array}{ccc}
& \\ & \downarrow \delta \\
& \\
& \\
Z & \rightarrow & \Sigma X
\end{array}
\quad \begin{array}{ccc}
& \\ & \downarrow \delta \\
& \\
& \\
& \\
\Sigma X' & \rightarrow & Z
\end{array}
\]

\textbf{Proposition 1.37.} The axioms $(TR_4)$, $(TR_4')$, $(TR_4'')$ are equivalent.

\section{1.5 Localisation of triangulated categories}

Roughly speaking localisation enlarges a ring (or a category) by adding inversions of certain ring elements (or morphisms). However strange things can happen here due to non-commutativity. Actually in all examples we are going to study the localisation is just a quotient of the original category.

The motivating example is the derived category of an abelian category, which is defined as a localisation of its homotopy category of chain complexes. For any additive category $A$, the homotopy category of chain complexes in $A$ is a triangulated category. The suspension is a shift here.

Mapping cones for chain maps behave as in homotopy theory. If $f : K \to L$ is a chain map, then

\[ K \xrightarrow{f} L \to C_f \to K[1] \]

is a mapping cone triangle. For $C^*$-algebras the contravariance of the functor $\text{Spaces} \to C^* - \text{alg}, X \mapsto C(X)$ causes confusion about direction of arrows.

If $F : A \to A'$ is additive functor, then the induced functor

\[ \text{Ho}(F) : \text{Ho}(A) \to \text{Ho}(A') \]

is exact - preserves suspensions and exact triangles.

\textit{Example 1.38.} Let $\Sigma : T \to T$ be a suspension functor, and

\[ A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1] \]

an exact triangle. The triangle


could be non-exact. To correct it we use an isomorphism

\[ \Sigma(A[1]) \xrightarrow{\text{id}} (\Sigma A)[1] \]

Passage to the derived category introduces homological algebra. The quasi-isomorphisms class, that is maps that induce an invertible maps on homology, is the class of morphisms which should be inverted in derived category.

\textit{Example 1.39.} The following map is a quasi-isomorphism

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \cdot 2 \\
0 & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z}/2 \\
\end{array}
\]
Definition 1.40. The localisation of a category $\mathcal{C}$ in a family of morphisms $S$ is a category $\mathcal{C}[S]$ together with a functor $F: \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that

1. $F(s)$ is invertible for all $s \in S$

2. $F$ is universal among functors with this property, that is if $G: \mathcal{C} \to \mathcal{C}'$ is another functor with $G(s)$ invertible for all $s \in S$, then there is a unique factorisation

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{C}[S^{-1}] \\
G \downarrow & & \downarrow \exists ! \\
\mathcal{C} & \xrightarrow{\approx} & \mathcal{C}[S^{-1}]
\end{array}
\]

In good cases there are some "commutation relations". We can introduce also a calculus of fractions. The pair

\[
\begin{array}{ccc}
A & \xrightarrow{\sim} & B \\
\downarrow s & & \downarrow f \\
\bullet & \xrightarrow{\sim} & \bullet
\end{array}
\]

can be rewritten as

\[
\begin{array}{ccc}
A & \xrightarrow{?} & \bullet \\
\downarrow g & & \downarrow f \\
B & \xrightarrow{\sim} & \bullet
\end{array}
\]

In good cases:

- For all $f \in \mathcal{C}, s \in S$ there exist $g, t$ such that $tf = gs \implies fs^{-1} = t^{-1}g$
- $S \circ S \subseteq S$ - compositon of morphisms in $S$ is in $S$.
- $s \cdot t \in S \implies t \in S$ - cancelation law.

In triangulated categories it is easier to specify which objects should become zero. Indeed for an exact triangle

\[
A \xrightarrow{f} B \to C \xrightarrow{} A[1]
\]

if $G$ is an exact functor, then $G(f)$ invertible implies $G(C) \approx 0$.

Definition 1.41. A class $\mathcal{N}$ of objects in a triangulated category $\mathcal{T}$ is called thick if it satisfies the following conditions

1. $0 \in \mathcal{N}$,
2. If $A \oplus B \in \mathcal{N}$ then $A, B \in \mathcal{N}$,
3. If the triangle $A \to B \to C \to A[1]$ is exact, and $A, B \in \mathcal{N}$, then $C \in \mathcal{N}$.

Notice that the object kernel \{ $A \in \mathcal{T} \mid G(A) \approx 0$ \} of an exact functor satisfies this.

Definition 1.42. Given a thick subcategory $\mathcal{N} \subseteq \mathcal{T}$ an $\mathcal{N}$-equivalence is a morphism in $\mathcal{T}$ which cone belongs to $\mathcal{N}$.

Denote

\[
\mathcal{T}/\mathcal{N} := \mathcal{T}[(\mathcal{N} - \text{equivalences})^{-1}]
\]

Theorem 1.43. Given a thick subcategory $\mathcal{N}$ in a (small) triangulated category $\mathcal{T}$, the $\mathcal{N}$-equivalences have a calculus of fractions, $\mathcal{T}/\mathcal{N}$ is again a triangulated category, and $\mathcal{T} \to \mathcal{T}/\mathcal{N}$ is an exact functor.
Definition 1.44. **Left orthogonal complement** of a class of objects $\mathcal{N}$ in $\mathcal{T}$

$$\mathcal{N}^\perp := \{P \in \mathcal{T} \mid T(P,N) = 0 \forall N \in \mathcal{N}\}$$

Definition 1.45. Two thick classes of objects $\mathcal{P}, \mathcal{N}$ in $\mathcal{T}$ are called **complementary** if

- $\mathcal{P} \subseteq \mathcal{N}^\perp$
- For all $A \in \mathcal{T}$ there is an exact triangle

$$P \rightarrow A \rightarrow N \rightarrow P[1], \quad P \in \mathcal{P}, \quad N \in \mathcal{N}.$$ 

Theorem 1.46. Let $(\mathcal{P}, \mathcal{N})$ be complementary. Then

1. $\mathcal{P} = \mathcal{N}^\perp, \mathcal{N} = \mathcal{P}^\perp$
2. the exact triangle $P \rightarrow A \rightarrow N \rightarrow P[1]$ with $P \in \mathcal{P}, \; N \in \mathcal{N}$ is unique up to canonical isomorphism and functorial in $\mathcal{A}$
3. the functors $\mathcal{T} \rightarrow \mathcal{P}, \; A \mapsto P, \; \mathcal{T} \rightarrow \mathcal{N}, \; A \mapsto N$ are exact.
4. $\mathcal{P} \rightarrow \mathcal{T}$ to $\mathcal{T}/\mathcal{N}$ and $\mathcal{N} \rightarrow \mathcal{T} \rightarrow \mathcal{T}/\mathcal{P}$ are equivalences of categories.

Example 1.47. Take $\text{Ho}(\mathcal{A}), \mathcal{A}$ abelian, $\mathcal{N} = \{\text{exact complexes}\}$. If $P \in \mathcal{A}$ is projective, then homotopy classes of chain maps $P \rightarrow C_\bullet$ (there is an inclusion $\mathcal{A} \hookrightarrow \text{Ho}(\mathcal{A})$) are in bijection with maps $P \rightarrow \text{Ho}(C_\bullet)$.

$$\begin{array}{ccc}
C_1 & \xrightarrow{d_1} & C_0 \\
\downarrow{f} & & \downarrow{d_0} \\
0 & \rightarrow & P \\
& \rightarrow & 0
\end{array}$$

Notice that $\mathcal{N}^\perp$ is always thick and closed under direct sums. Subcategories with both properties are called localising.

Example 1.48. Let $P_0, P_1$ be projective in $\mathcal{A}$, and $f: P_1 \rightarrow P_0$. Then its cone

$$C_f := (\ldots \rightarrow 0 \rightarrow P_1 \xrightarrow{f} P_0 \rightarrow 0 \rightarrow \ldots)$$

Theorem 1.49 (Boekstadt-Neemann). Suppose that $\mathcal{A}$ is abelian category with enough projectives and countable direct sums. Let $\mathcal{N} \subseteq \text{Ho}(\mathcal{A})$ be the full subcategory of exact chain complexes, and let $\mathcal{P}$ be the localising subcategory generated by the projective objects of $\mathcal{A} \hookrightarrow \text{Ho}(\mathcal{A})$. Then $(\mathcal{P}, \mathcal{N})$ are complementary.

The functor $P: \text{Ho}(\mathcal{A}) \rightarrow \mathcal{P}$ replaces a module by a projective resolution of the module

$$P(M) = (\ldots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \rightarrow \ldots)$$

Example 1.50. Let $\mathcal{T} = \text{KK}, \mathcal{N} = \{A \in \text{KK} \mid K_* (A) = 0\}$. Then $\mathcal{C} \in \mathcal{N}^\perp$ because $\text{KK}_d(\mathcal{C}, A) = K_*(A) = 0$ for $A \in \mathcal{N}$. Let $\mathcal{B}$ be the localising subcategory generated by $\mathcal{C}$.

Theorem 1.51. $(\mathcal{B}, \mathcal{N})$ are complementary.
$P$: KK $\rightarrow B$ replaces a separable C*-algebra by one in the bootstrap class with the same K-theory.

Let $(\mathcal{P}, \mathcal{N})$ be complementary subcategories. Then

1. $\mathcal{P} = \mathcal{N}^\perp$. From the assumption $\mathcal{P} \subseteq \mathcal{N}^\perp$ and take $A \in \mathcal{N}^\perp$ and embed it into an exact triangle

$$\begin{array}{ccc}
P \rightarrow A & \rightarrow N & \rightarrow P[1] \\
\end{array}$$

There is a splitting $A \rightarrow P$, so $A$ is a direct summand of $P$, hence $A \in \mathcal{P}$, because $\mathcal{P}$ is thick.

2. Let $A, A' \in T$, $f: A \rightarrow A'$. Then there is a map of exact triangles

$$\begin{array}{ccc}
P & \rightarrow A & \rightarrow N & \rightarrow P[1] \\
\downarrow f & & \downarrow \Sigma & \\
P' & \rightarrow A' & \rightarrow N' & \rightarrow P'[1] \\
\end{array}$$

with $P, P' \in \mathcal{P}$, $N, N' \in \mathcal{N}$.

We use long exact sequence

$$\cdots \rightarrow T(P, N') \rightarrow T(P, P') \xrightarrow{\sim} T_0(P, A') \rightarrow T(P, N') \rightarrow \cdots$$

to get $P \xrightarrow{P_f} P'$ in the diagram

$$\begin{array}{ccc}
P & \rightarrow A & \rightarrow N & \rightarrow P[1] \\
\downarrow P_f & \downarrow f & \downarrow \Sigma P_f & \\
P' & \rightarrow A' & \rightarrow N' & \rightarrow P'[1] \\
\end{array}$$

Then use (TR3) to extend $(f, P_f)$ to a morphism of exact triangles by $N \xrightarrow{N_f} N'$, which is unique making the diagram

$$\begin{array}{ccc}
A & \rightarrow N \\
f & \downarrow N_f & \\
A' & \rightarrow N' \\
\end{array}$$

commute.

3. $\mathcal{P}, \mathcal{N}$ are exact.

From (TR1) there is $X$ in the exact triangle

$$P_A \rightarrow P_B \rightarrow X \rightarrow P_A[1]$$

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From (TR3) we can find \( X \xrightarrow{f} C \) in the diagram

\[
\begin{array}{ccc}
P_A & \rightarrow & P_B \\
\downarrow{\pi_A} & & \downarrow{\pi_B} \\
A & \rightarrow & B \\
\downarrow & & \downarrow \\
N_A & \rightarrow & N_B \\
\downarrow & & \downarrow \\
\end{array}
\quad
\begin{array}{ccc}
X & \rightarrow & P_{A[1]} \\
\downarrow{f} & & \downarrow \\
P_{A[1]} & \rightarrow & P_{B[1]}[1] \\
\end{array}
\quad
\begin{array}{ccc}
P_A & \rightarrow & X \quad \text{Cone}(f) \rightarrow N_A[1] \\
\downarrow & & \downarrow \\
\end{array}
\quad
\begin{array}{ccc}
P_A & \rightarrow & P_B \\
\downarrow & & \downarrow \\
P_{A[1]} & \rightarrow & P_{B[1]}[1] \\
\end{array}
\quad
\begin{array}{ccc}
P_A & \rightarrow & X \quad \text{Cone}(f) \rightarrow N_A[1] \\
\downarrow & & \downarrow \\
\end{array}
\quad
\begin{array}{ccc}
P_A & \rightarrow & P_B \\
\downarrow & & \downarrow \\
P_{A[1]} & \rightarrow & P_{B[1]}[1] \\
\end{array}
\]

Thus \( X = P_C \) and \( \text{Cone}(f) = N_C \) and \( f \) must be the canonical map \( P_C \rightarrow C \).

\( T_s(Q, \pi_A) \) and \( T_s(Q, \pi_B) \) are invertible because \( N_A \in \mathcal{N} \), \( N_B \in \mathcal{N} \). Now we use the five lemma for

\[
\begin{array}{ccc}
T(Q, P_A) & \rightarrow & T(Q, P_B) \\
\downarrow{\cong} & & \downarrow{\cong} \\
T(Q, A) & \rightarrow & T(Q, B) \\
\end{array}
\quad
\begin{array}{ccc}
T(Q, X) & \rightarrow & T(Q, P_{A[1]}) \\
\downarrow{\cong} & & \downarrow{\cong} \\
T(Q, C) & \rightarrow & T(Q, P_{B[1]}) \\
\end{array}
\quad
\begin{array}{ccc}
T(Q, P_A) & \rightarrow & T(Q, P_B) \\
\downarrow{\cong} & & \downarrow{\cong} \\
T(Q, A) & \rightarrow & T(Q, B) \\
\end{array}
\quad
\begin{array}{ccc}
T(Q, P_{A[1]}) & \rightarrow & T(Q, P_{B[1]}) \\
\downarrow{\cong} & & \downarrow{\cong} \\
T(Q, A[1]) & \rightarrow & T(Q, B[1]) \\
\end{array}
\]

There is an isomorphism \( P_{A[1]} \simeq P_{A[1]} \).

For an exact triangle

\[
P \xrightarrow{u} A \xrightarrow{v} N \xrightarrow{w} P[1]
\]
the triangle

\[
\]
is exact.

We have seen along the way that \( T(Q, P_A) \simeq T(Q, A) \) for all \( Q \in \mathcal{P} \), which means that the functor \( P: \mathcal{T} \rightarrow \mathcal{P} \) is right adjoint to the embedding \( \mathcal{P} \hookrightarrow \mathcal{T} \).

Define \( \mathcal{T}' \) as the category with the same objects as \( \mathcal{T} \) and \( T'(A, B) := T(P_A, P_B) \). Let \( F: \mathcal{T} \rightarrow \mathcal{T}' \) be the functor that is the identity on objects and \( P \) on morphisms. This satisfies the universal property of \( T[(\mathcal{N} - \text{equivalences})^{-1}] \). Notice that \( P_A \simeq A \) if \( A \in \mathcal{P} \). Also \( P_A \rightarrow A \) is an \( \mathcal{N} \)-equivalence.

If the triangle

\[
A \xrightarrow{P_u} B \xrightarrow{P_v} C \xrightarrow{P_w} A[1]
\]
is exact in \( \mathcal{T}' \), then the triangle

\[
P_A \xrightarrow{P_u} P_B \xrightarrow{P_v} P_C \xrightarrow{P_w} P_{A[1]}
\]
is exact in \( \mathcal{T} \).

\( P \) maps \( \mathcal{N} \)-equivalences to isomorphisms because \( P(A) = 0 \) for \( A \in \mathcal{N} \). If \( G \) maps \( \mathcal{N} \)-equivalences to isomorphisms we get

\[
\begin{array}{ccc}
G(P_A) & \rightarrow & G(P_B) \\
\downarrow{\cong} & & \downarrow{\cong} \\
G(A) & \rightarrow & G(B)
\end{array}
\]

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so $T'(A,B)$ gives a map $G(A) \rightarrow G(B)$.

Let $T$ be triangulated and monoidal, and let $\mathcal{P}, \mathcal{N}$ be thick subcategories with $\mathcal{P} \otimes T \subseteq \mathcal{P}$, $\mathcal{N} \otimes \mathcal{P} \subseteq \mathcal{N}$. If there is an exact triangle

$$P \rightarrow 1 \rightarrow N \rightarrow P[1],$$

where $1$ is the tensor unit, $P \in \mathcal{P}$, $N \in \mathcal{N}$, and $\mathcal{P} \subseteq \mathcal{N}^r$, then $(\mathcal{P}, \mathcal{N})$ are complementary. Also for an arbitrary $A$ the triangle

$$P \otimes A \rightarrow 1 \otimes A \rightarrow N \otimes A \rightarrow P \otimes A[1],$$

is exact.

We expect that $KK^G$ has a (symmetric) monoidal structure also if $G$ is a quantum group.

**Example 1.52.** Let $G$ be finite group, $A, B$ algebras with $G$-coaction (grading). Then $A \otimes B$ carries a diagonal coaction

$$(A \otimes B)_g = \bigoplus_{h \in G} A_h \otimes B_{h^{-1}g}$$

We want to equip $A \otimes B$ with a multiplication that is equivariant for the canonical coaction of $G$ on $A \otimes B$. The usual product does not work, because if $a \in A_h$, $b \in B_g$, then $a \cdot b = b \cdot a \in (A \otimes B)_{hg}$ but we need $b \cdot a \in (A \otimes B)_{gh}$. We must therefore impose a commutation relation that is non-trivial. We define

$$b_g \cdot a_h := \alpha_g(a_h) \cdot b_g, \text{ for } a_h \in A_h, b_g \in B_g,$$

where $\alpha_g \colon A \rightarrow A$ for $g \in G$ is some linear map. Associativity dictates that $\alpha_g(a_1 \cdot a_2) = \alpha_g(a_1) \alpha_g(a_2)$, and $\alpha_g \alpha_{g_2} = \alpha_{g_1 g_2}$. It is natural to require also $\alpha_1 = \text{id}_A$, so that $\alpha$ is an action of $G$ on $A$ by algebra automorphisms. Finally covariance dictates that $\alpha_g(A_h) \subseteq A_{ghg^{-1}}$ for all $g, h \in G$.

The extra structure $\alpha$ should always exist on a stabilisation $E_A := \text{End}(A \otimes \mathbb{C}[G])$ with the coaction of $G$ induced by the tensor product coaction on $A \otimes \mathbb{C}[G]$. $A_h \otimes \langle \delta_g \rangle \langle \delta_1 \rangle$ maps $(A \otimes \mathbb{C}[G])_x$ to $A_{x^{-1}h} \otimes \mathbb{C}[G]_g \subseteq (A \otimes \mathbb{C}[G])_{x^{-1}hg}$, hence

$$(E_A)_g = \sum_{x, y, z \in G, x^{-1}yz = g} A_y \otimes \langle \delta_z \rangle \langle \delta_x \rangle$$

Let $G$ act on $A \otimes \mathbb{C}[G]$ by the regular representation. This induces an action $\alpha \colon G \times E_A \rightarrow E_A$ by conjugation. We check that if $x^{-1}yz = h$, then

$$\alpha_g(A_y \otimes \langle \delta_z \rangle \langle \delta_x \rangle) = A_y \otimes \langle \delta_{zg^{-1}} \rangle \langle \delta_{xyg^{-1}} \rangle \in (E_A)_{gxy^{-1}zg^{-1}} = (E_A)_{ghg^{-1}}$$

Thus $E_A \otimes B$ carries a canonical algebra structure.

Even in homological algebra, in $\text{Ho}(R - \text{Mod})$ it is not obvious that the exact chain complexes are part of a complementary pair.

$$\text{Der}(R - \text{Mod}) := \text{Ho}(R - \text{Mod})/(\text{exact chain complexes})$$

Recall $(\mathcal{L}, \mathcal{N})$ is complementary if

- $\text{Hom}(\mathcal{L}, \mathcal{N}) = 0$

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• For all $A \in T$ there exist an exact triangle

$$L \to A \to N \to L[1]$$

With $L \in \mathcal{L}$, $N \in \mathcal{N}$.

We will explain a general method for doing homological algebra in a triangulated categories that also, eventually solves this problem.

Assume we want to understand a triangulated category $T$. As a probe to explore it, we use some homological functor $F : T \to \mathcal{A}$, where $\mathcal{A}$ is some abelian category.

**Examples 1.53.**

- $T = \text{Ho}(\mathcal{A})$, $\mathcal{A}$ an abelian category, and $F$ is a homology functor $\text{Ho}(\mathcal{A}) \to \mathcal{A}^Z$.

- $T = \text{KK}$, $F = K_* : \text{KK} \to \text{Ab}^{Z/2}$.

- $T = \text{KK}^{(C, \Delta)}$, where $(C, \Delta)$ is a quantum group, $F = K_* : \text{KK} \to \text{Ab}^{Z/2}$.

In the examples above, the target category has its own translation (suspension) automorphism, and $F$ intertwines these translation automorphisms, we call $F$ stable if this happens.

Actually, all the relevant information about $F$ is contained in its morphism-kernel $(\ker F)(A, B) := \{ \phi : A \to B \mid F(\phi) = 0 \}$

This is a finer invariant than the object kernel. $\ker F$ is called a homological ideal. Using homological ideal we can carry over various notions from homological algebra to our category $T$.

**Definition 1.54.** Let $(C_n, d_n)$ be a chain complex in $T$. We call it $\ker F$-exact in degree $n$ if $F(C_{n+1}) \to F(C_n) \to F(C_{n-1})$ is exact at $F(C_n)$

Here $F$ is exact, but it depends only on $\ker F$, so we call it $\ker F$-exact.

**Definition 1.55.** An object $A \in T$ is $\ker F$-projective if the functor $T(A, -)$ maps $\ker F$-exact chain complexes in $T$ to exact chain complexes.

Denote $\mathcal{J} := \ker F$.

**Lemma 1.56.** The following statements are equivalent

1. an object $A \in T$ is $\mathcal{J}$-projective

2. for all $f \in \mathcal{J}(B, C)$ the map $T(A, B) \xrightarrow{f_\ast} T(A, C)$ vanishes

3. for all $C \in T$ $\mathcal{J}(A, C) = 0$

**Definition 1.57.** A projective resolution of $A \in T$ is a $\mathcal{J}$-exact chain complex

$$\ldots \to P_2 \to P_1 \to P_0 \to A \to 0 \to \ldots$$

with $P_i$ $\mathcal{J}$-projective.

Now we can ask the following questions:
• What are the projective objects in examples?
• Are there many of them? That is does every object have a $J$-projective resolution?

We use (partially defined) left adjoints to decide this. Let $F: T \to A$ be stable homological with $\ker F = J$. Its left adjoint $F^\leftarrow$ is defined on $B \in A$ if there is $B' := F^+(B)$ with $T(B', D) \simeq A(B, F(D))$ for all $D \in T$, natural in $D$. This defines a functor on a subcategory of $A$.

The functor $T(F^+(B), -)$ factors as follows

$$ T \xrightarrow{F} A \xrightarrow{A(B, -)} \text{Ab} \xrightarrow{A(B, F(D))} $$

and therefore vanishes on $J = \ker F$.

**Examples 1.58.**

1. Let $T = \text{Ho}(A), F = H_* : \text{Ho}(A) \to A^Z$. Assume that $A$ has enough projectives. Recall that if $P \in A$ is projective, then $T(P, C_\bullet) = A(P, H_*(C_\bullet))$. Thus $H_*^+$ is defined on projective objects of $A$ or $A^Z$ and it produces a chain complex with vanishing boundary map.

2. Let $T = KK, F = K_* : KK \to \text{Ab}^{Z/2}$. Because

$$ KK(C, A) = K_*(A) = \text{Hom}(Z, K_*(A)) $$

we have

$$ K_*^+(Z[0]) = \mathbb{C} $$

$$ K_*^+(Z[1]) = \mathbb{C}[1] = C_0(\mathbb{R}) $$

Left adjoints commute with direct sums, hence $K_*^+$ is defined on free $Z/2$ graded abelian groups.

3. Let $T = KK^Z$ be an equivariant $KK$-theory for integers, and $F : KK^Z \to \text{Ab}^{Z/2}, F(A, \alpha) = K_*(A)$. If $A \in KK, b \in KK^Z$ then

$$ KK^Z(C_0(Z) \otimes A, B) = KK(A, B) $$

More generally, if $H \subseteq G$ is an open subgroup, then

$$ KK^G(\text{Ind}_H^G A, B) \simeq KK^H(A, \text{Res}_G^H B) $$

Here we had $G = Z, H = \{1\}$.

Since $(F \circ G)^+ = G^+ \circ F^+$. $F^+$ is defined on all free $Z/2$-graded abelian groups, and given by

$$ F^+(Z[0]) = C_0(G), \quad (G = Z) $$

**Proposition 1.59.** Let $F: T \to A$ be a stable homological functor whose left adjoint is defined on all projective objects of an abelian category $A$. If $A$ has enough projectives, then there are enough $\ker F^+$-projective objects in $T$, and any $\ker F^+$-projective object is a retract of $F^+(B)$ for some projective object $B \in A$. 

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Proof. Let $D \in T$, we need $B \in \mathcal{A}$ projective and a morphism $\pi \in T(F^+(B), D)$ such that $F(\pi)$ is an epimorphism. This is the beginning of a recursive construction of a projective resolution. We have

$$T(F^+(B), D) \simeq \mathcal{A}(B, F(D))$$

$$\rho^* \leftarrow \rho$$

We claim that $F(\rho^*)$ is an epimorphism. There is a commutative diagram

$$\begin{array}{ccc}
FF^+(B) & \xrightarrow{F(\rho^*)} & F(D) \\
\downarrow \varepsilon_B & & \downarrow \rho \\
B & \xrightarrow{\rho} & F(D)
\end{array}$$

where $\varepsilon: \text{Id} \to FF^+$ is a unit of adjointness.

Once we have $\mathcal{J}$-projective resolution, we get $\mathcal{J}$-derived functors. The question is how to compute them?

There are three conditions:

1. $F \circ F^+ = \text{id}_{\text{Proj}_A}$
2. $\text{Proj}_J F \to \text{Proj}_A$
3. \[
\left\{ \begin{array}{c}
\mathcal{J} - \text{projective resolutions of } D \in T \\
\text{up to isomorphism}
\end{array} \right\} \simeq \left\{ \begin{array}{c}
\text{projective resolutions of } F(D) \\
\text{up to isomorphism}
\end{array} \right\}
\]

Example 1.60. Let $D \in \text{KK}$, and there is a free resolution of its K-theory

$$\ldots \to 0 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} K_*(D) \to 0$$

Then

$$\text{KK}(K^+_*(P_1), K^+_*(P_0)) = \text{Hom}_{\text{Ab}^{Z/2}}(P_1, P_0)$$

By (2) we can lift $d_1$ to $\tilde{d}_1: K^+_*(P_1) \to K^+_*(P_0)$

$$\text{KK}(K^+_*(P_0), D) \simeq \text{KK}(P_0, K_*(D))$$

$$\tilde{d}_0 \mapsto d_0$$

Then

$$0 \to K^+_*(P_1) \to K^+_*(P_0) \to 0 \to 0$$

is an $\mathcal{J}$-projective resolution, $\mathcal{J} = \ker(K_*)$. Both $K^+_*(P_0)$ and $K^+_*(P_1)$ are direct sums of $C$ and $C_0(\mathbb{R})$, and

$$K_*(K^+_*(P_j)) = P_j$$

Hence we have lifted a projective resolution in $\text{Ab}^{Z/2}$ to one in KK.

In the nice case where (2) and hence (1) and (3) hold, the derived functors with respect to $\mathcal{J}$ are the same as derived functors in the abelian category $\mathcal{A}$ because resolutions are the same.
Proposition 1.61. Assuming (1), any homological functor, \( H : T \to C \) induces a right-exact functor \( H : A \to C \) and \( \mathbb{L}_p H = \mathbb{L}_p H \circ F \)

\[
\text{Ext}^n_{(T,J)}(D, E) \simeq \text{Ext}_{A}^n(F(D), F(E))
\]

Example 1.62. Because

\[
\text{Ext}^n_{(KK, \ker(K_*))}(D, E) = \text{Ext}_{\mathbb{A}b/\mathbb{Z}}^n(K_*(D), K_*(E))
\]

for all \( n \geq 1 \), we have

\[
\text{Ext}^0_{(KK, \ker(K_*))} = \text{Hom}, \quad \text{Ext}^n_{(KK, \ker(K_*))} = 0
\]

There is a canonical map

\[
T(D, E)/J(D, E) \to \text{Ext}^0_{(T,J)}(D, E)
\]

The general feature is that \( J \) acts by 0 on all derived functors.

Definition 1.63. Let \( D \in T \), \( (P_n, \partial_n) \) be an \( J \)-projective resolution of \( D \). Then \( \text{Ext}^n_{(T,J)}(D, E) \) is the \( n \)-th cohomology of

\[
\ldots \leftarrow sT(P_n, E) \leftarrow T(P_{n-1}, E) \leftarrow \ldots \leftarrow T(P_0, E) \leftarrow 0
\]

For example

\[
\text{Ext}^0_{T,J} = \ker(T(P_0, E) \to T(P_1, E))
\]

\[
P_1 \longrightarrow P_0 \longrightarrow D \longrightarrow 0 \quad \downarrow \alpha
\]

\[
\quad \downarrow \alpha
\]

\[
E
\]

Assume we want to understand a triangulated category \( T \), that may have nothing to do with algebra, using the tools from homological algebra. We have been able to define projective resolutions and thus derived functors. How to achieve \( F \circ F^+ = \text{id} \)? Is there abelian category that describes the derived functors?

Definition 1.64. Let \( J \subseteq T \) be a homological ideal. A stable homological functor \( F : T \to A \) with \( \ker F = J \) is called universal (for \( J \)) if any other stable homological functor \( H : T \to A' \) with \( \ker H \supseteq J \) factors through \( F \) uniquely up to equivalence.

Theorem 1.65. If the left adjoint \( F^+ \) is defined on all projective objects and \( F \circ F^+ = \text{id}_{\text{Proj}_A} \) then \( F \) is universal for \( ker F \).

Conversely, if \( ker F \) has enough projectives, and \( F \) is universal, then \( F^+ \) is defined on all projective objects and \( F \circ F^+ = \text{id}_{\text{Proj}_A} \).

Proof. Assume we have a functor \( H : T \to C \)

\[
\begin{array}{ccc}
T & \xrightarrow{F} & A \\
\downarrow H & & \downarrow \Pi \\
C & & C
\end{array}
\]

We want to prove that there is a unique \( \overline{H} : A \to C \). There is a following sequence of functors

\[
A \to \text{Ho}(\text{Proj}_A) \simeq \text{Ho}(\text{Proj}_J) \subseteq \text{Ho}(T) \xrightarrow{H} \text{Ho}(C) \xrightarrow{\text{Ho}\alpha} C
\]

First functor is taking the projective resolution, on objects \( B \mapsto (P_n, \alpha_n) \).

\[
\square
\]
**Example 1.66.** The functor

\[
\text{KK}^\mathbb{Z} \to \text{Ab}^{\mathbb{Z}/2} \\
(D, \alpha) \mapsto K_*(D)
\]

is not universal. The universal functor \( \tilde{F} \) here is defined on all projective objects and satisfies \( \tilde{F} \circ \tilde{F}^+ = \text{id}_{\text{Proj}_{\text{Ab}}^{\mathbb{Z}/2}} \). Notice that the \( \mathbb{Z} \)-action on \( D \) induces an action on \( K_*(D) \). We enrich \( F \) to a functor

\[
\tilde{F} : \text{KK}^\mathbb{Z} \to \text{Mod}(\mathbb{Z}[\mathbb{Z}])^{\mathbb{Z}/2} \\
\tilde{F}(D) := \text{KK}_*(\mathcal{C}, D) = \text{KK}^{\mathbb{Z}}(C_0(\mathbb{Z}), (D, \alpha))
\]

Then \( \ker \tilde{F} \) and \( \tilde{F} \) is universal. Furthermore

\[
\text{Hom}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}], \tilde{F}(D)) = \tilde{F}(D)
\]

Thus \( \tilde{F}^+(\mathbb{Z}[\mathbb{Z}]) = C_0(\mathbb{Z}) \) and \( \tilde{F} \circ \tilde{F}^+ = \mathbb{Z}[\mathbb{Z}] \).

**Example 1.67.** Take the homology functor

\[
F = H_* : (R - \text{Mod}) \to \text{Ab}^{\mathbb{Z}}
\]

Passing from \( F \) to the universal functor for \( \ker F \) reconstructs \( H_* : \text{Ho}(R - \text{Mod}) \to (R - \text{Mod})^{\mathbb{Z}} \). The left adjoint \( H_*^\nu \) is defined on projective modules, and \( H_* \circ H_*^\nu = \text{id} \).

**Example 1.68.** Let \((C, \Delta)\) be a discrete quantum group, \( T = \text{KK}^{(C, \Delta)} \), \( F(A, \Delta_A) = K_*(A) \) for a separable C*-algebra with coaction \( \Delta_A : A \to \mathcal{M}(A \otimes C) \).

\( F \) is a poor invariant - it forgets too much. Say \( C = C^*(G) \) for finite \( G \). Then

\[
\text{KK}^{(C, \Delta)}(C \otimes A, B) \simeq \text{KK}(A, B)
\]

The left adjoint \( F^+ \) is defined on free abelian groups. From Baaj-Skandalis duality

\[
\text{KK}^{(C, \Delta)}(A, B) = \text{KK}^{(\hat{C}, \Delta)}(A \rtimes \hat{C}, B \rtimes \hat{C})
\]

\[
A \rtimes \hat{C} \rtimes C \simeq A \otimes \mathcal{K}(\mathcal{H}_C) \simeq A
\]

There turns out to be a canonical \( \text{Rep}(\hat{C}) \)-module structure on \( K_*(A \rtimes C) =: K^\hat{C}_*(A) \).

In Baaj-Skandalis duality example

\[
\text{KK}^\mathbb{Z}(A, B) \simeq \text{KK}^{U(1)}(A \rtimes \mathbb{Z}, B \rtimes \mathbb{Z})
\]

Let \( T \) be a triangulated category (with direct sums), and \( F : T \to A \) be a stable homological functor into some abelian category (commuting with direct sums). The left adjoint of \( F \) is defined on all projective objects in \( A \).

**Examples 1.69.**

- \( T = \text{Ho}(\tilde{A}), F : T \to \tilde{A}^{\mathbb{Z}}, F(C_\bullet) = H_*(C_\bullet) \)
- \( T = \text{KK}, F : \text{KK} \to \text{Ab}^{\mathbb{Z}/2}, F(B) = K_*(B) \)
- \( T = \text{KK}^\mathbb{Z}, F : \text{KK} \to \text{Ab}^{\mathbb{Z}/2}, F(B, \beta) = K_*(B), F^+(\mathbb{Z}) = C_0(\mathbb{Z}) \) with free action of \( \mathbb{Z} \)
Let $L$ be the smallest subcategory of $\mathcal{T}$ that is thick, contains all ker $F$-projective objects, and is closed under direct sums. Let $\mathcal{N} = \{ A \in \mathcal{T} \mid F(A) = 0 \}$. Then if $L \in \mathcal{L}$, $N \in \mathcal{N}$ we have $\mathcal{T}(L,N) = 0$ because it holds if $L$ is ker $F$-projective, and $\{ A \mid \mathcal{T}(A,N) = 0 \}$ is localising. For $\mathcal{L}, \mathcal{N}$ to be complementary, we need that any $B \in \mathcal{T}$ can be embedded in an exact triangle

$$L \to B \to N \to L[1], \quad L \in \mathcal{L}, \ N \in \mathcal{N}$$

**Theorem 1.70.** If $F: \mathcal{T} \to \mathcal{A}$ commutes with direct sums and $\mathcal{T}$ has enough ker $F$-projectives, then $(\mathcal{L}, \mathcal{N})$ are complementary.

**Example 1.71.** For $K_*$ on $KK$

$$\mathcal{L} = \langle \mathbb{C} \rangle, \quad \mathcal{N} = \{ B \in KK \mid K_*(B) = 0 \}$$

**Example 1.72.** For $K_*$ on $KK^\mathbb{Z}$

$$\mathcal{L} = \langle C_0(\mathbb{Z}) \rangle = \{ (B, \beta) \in KK^\mathbb{Z} \mid B \text{ is the bootstrap class} \}$$

The inclusion $\subset$ is obvious, and $\supset$ is closely related to the Pimsner-Voiculescu sequence and the Baum-Connes conjecture for $\mathbb{Z}$. We will give a sketch of the proof.

Take $(B, \beta) \in KK^\mathbb{Z}$. Look at the extension

$$C_0(\mathbb{R}, B) \hookrightarrow C_0(\mathbb{R} \cup \{+\infty\}, B) \twoheadrightarrow B$$

Here we have an action of $\mathbb{Z}$ on $\mathbb{R}$ by translation. This extension does not have an equivariant completely positive section. But an argument by Baaj-Skandalis shows that it yields an extension triangle nevertheless.

$$C_0(\mathbb{Z} \times (0,1)) \twoheadrightarrow C_0(\mathbb{R}, B) \twoheadrightarrow C_0(\mathbb{Z}, B)$$

If $B \in \langle \mathbb{C} \rangle$, then $C_0(\mathbb{Z}, B)$ and $C_0(\mathbb{Z} \times (0,1))$ belong to $\langle C_0(\mathbb{Z}) \rangle$, hence so does $C_0(\mathbb{R}, B)$.

**Theorem 1.73.** $C_0((-\infty, \infty], B) \simeq 0$ in $KK^\mathbb{Z}$ with diagonal action.

This is where the work has to be done. More generally, if $(B, \beta) \in KK^\mathbb{Z}$ satisfies $B \simeq 0$ in $KK$, then $(B, \beta) \simeq 0$ in $KK^\mathbb{Z}$. Equivalently if $f \in KK^\mathbb{Z}(B_1, B_2)$ is invertible in $KK(B_1, B_2)$, then $f$ is invertible in $KK^\mathbb{Z}$.

More generally we can replace $\mathbb{Z}$ by any torsion-free (that is without compact subgroups) a-T-menable locally compact group. It is implied by the proof of the Baum-Connes conjecture by Higson and Kasparov.

The full proof of the fact that $(\mathcal{L}, \mathcal{N})$ are complementary is in Ralf Meyer, "Homological algebra in triangulated category", part II. We will prove a weaker fact, that is $(\mathcal{N}^+, \mathcal{N})$ are complementary. The proof uses phantom tower (maps in ker $F$ are called phantom maps).

**Definition 1.74.** Let $B \in \mathcal{T}$. Phantom tower is a diagram of the form

$$B \xrightarrow{i_0} N_0 \xrightarrow{i_1} N_1 \xrightarrow{i_2} N_2 \xrightarrow{i_3} N_3 \xrightarrow{i_4} \ldots$$

$\xleftarrow{P_0}$ $\xleftarrow{P_1}$ $\xleftarrow{P_2}$ $\xleftarrow{P_3}$

(1.6)
where all $P_n$ are ker $F$-projective, $i^{n+1}_n \in \ker F$, and all triangles

\[
\begin{array}{ccc}
N_n & \xrightarrow{i^{n+1}_n} & N_{n+1} \\
\downarrow & & \downarrow \\
P_n & & 
\end{array}
\]

are exact. This means that the maps $N_{n+1} \rightarrow P_{n-1}$ are of degree 1, that is actually $N_{n+1} \rightarrow P_{n-1}[1]$. The bottom row

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \ldots$$

is a chain complex with differential of degree 1.

**Proposition 1.75.** Given a phantom tower (1.6), the complex

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \ldots$$

is a projective resolution. Conversely, any projective resolution embeds uniquely in a phantom tower.

**Proof.** The sequence

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \ldots$$

is ker $F$-exact. We know that

$$F_{*+1}(N_{j+1}) \rightarrow F_*(P_j) \rightarrow F_*(N_j)$$

is a short exact sequence because $F(i_{j+1}^j) = 0$. The Yoneda product of these extensions is the chain complex

$$F(B) \leftarrow F(P_0) \leftarrow F(P_1) \leftarrow \ldots$$

This is exact as a Yoneda product of extensions. Now take a projective resolution

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \ldots$$

Recursively construct $N_j$ starting with $N_0 = B$. Now embed $N_j \leftarrow P_j$ in an exact triangle

$$P_j \rightarrow N_j \xrightarrow{i_j^{j+1}} N_{j+1} \rightarrow P_{j+1}[1].$$

Induction assumption tells that $N_j \leftarrow P_j$ is ker $F$-epimorphism, that is $F(P_j) \rightarrow F(N_j)$ is an epimorphism. Then $F(i_{j+1}^j) = 0$ because $F$ is homological. Now we must lift the boundary map $P_{j+1} \rightarrow P_{j+1}[1]$ to a map $P_{j+1} \rightarrow N_{j+1}$. and check that then it is ker $F$-epimorphism.

In the sequence

$$T(P_{j+1}, N_j) \rightarrow T(P_{j+1}, N_{j+1}) \rightarrow T(P_{j+1}, P_{j+1}[1]) \rightarrow T(P_{j+1}, N_{j+1}[1])$$

the first map is zero, because $P_{j+1}$ is projective and $i_{j+1}^j$ is phantom.

Because the composition

$$P_{j+1} \rightarrow P_j[1] \rightarrow P_{j-1}[2]$$

vanishes, the boundary map goes to 0 in $T(P_{j+1}, N_{j+1}[1])$, hence comes from $T(P_{j+1}, N_{j+1})$.

Now routine check that it is an epimorphism.
Now we will prove that for any $B \in \mathcal{T}$ there is $N \in \mathcal{N}$ and a map $f: B \to N$ such that
\[
T_*(N, M) \to T_*(B, M)
\]
is invertible for all $M \in \mathcal{N}$. Then $B \mapsto N$ is a functor $\mathcal{T} \to \mathcal{N}$ that is left adjoint to the embedding functor $\mathcal{N} \to \mathcal{T}$. We let $N$ to be the homotopy direct limit of the phantom tower.
\[
\bigoplus_j N_j \xrightarrow{\text{id}-S} \bigoplus_j N_j \to \text{holim} N_j \to \bigoplus_j N_j[1], \quad S = \bigoplus_j i_j^{j+1}
\]
Since $F$ commutes with direct sums, and $i_j^{j+1} \in \ker F$, $F(S) = 0$. Therefore $F(\text{id}-S) = F(\text{id})$ is invertible so that $F(\text{holim} N_j) = 0$.

Let $M \in \mathcal{N}$. Then $T_*(P_j, M) = 0$ because $P_j$ is ker $F$-projective. Therefore $i_j^{j+1}$ induces an isomorphism
\[
T_*(N_{j+1}, M) \cong T_*(N_j, M)
\]
There is an extension
\[
\begin{array}{ccc}
\lim^1 T_*(\text{holim} N_j, M) & \xrightarrow{T_*} & T_*(N_j, M) \\
\downarrow & & \downarrow \\
T_*(N_0, M) & \xrightarrow{T_*} & T_*(B, M)
\end{array}
\]

### 1.6 Index maps in K-theory and K-homology

Consider the following extension of $C^*$-algebras
\[
I \xrightarrow{i} E \xrightarrow{p} Q
\]
There are long exact sequences in K-theory and in K-homology:
\[
\begin{array}{ccc}
K_0(I) & \xrightarrow{\partial} & K_1(E) \\
\uparrow & & \uparrow \\
K_1(Q) & \leftarrow K_1(E) & \leftarrow K_1(I)
\end{array}
\]
\[
\begin{array}{ccc}
K^0(I) & \xrightarrow{\delta} & K^0(E) \\
\uparrow & & \uparrow \\
K^1(I) & \leftarrow K^1(E) & \leftarrow K^1(Q)
\end{array}
\]
and we have pairings between K-theory and K-homology. We will prove that
\[
-\langle \partial(x), y \rangle = \langle x, \delta(y) \rangle, \quad x \in K_1(Q), \ y \in K^0(I)
\]
We will use only formal properties of the boundary maps.

**Theorem 1.76.** Let
\[
\begin{array}{c}
\partial: K_1(Q) \to K_0(I) \\
\delta: K^0(I) \to K^1(Q)
\end{array}
\]
be natural for morphisms of extensions. Then there is \( \varepsilon \in \{\pm 1\} \) such that
\[
(\partial(x), y) = \varepsilon \langle x, \delta(y) \rangle
\]
for all extensions and all \( x \in K_1(Q), y \in K^0(I) \).

Remark 1.77. The sign \( \varepsilon \) is fixed by looking at the extension
\[
\mathcal{K} \hookrightarrow T \rightarrow C(S^1)
\]
and the generators of \( K_1(C(S^1)) = \mathbb{Z}, K^0(\mathcal{K}) = \mathbb{Z} \).
\[
[\mathcal{K} \hookrightarrow T \rightarrow C(S^1)] \in K^1(C(S^1)) \simeq \text{Hom}(K_1(C(S^1))) \simeq \mathbb{Z}
\]
\[
[\mathcal{K} \hookrightarrow T \rightarrow C(S^1)] \mapsto -1 \in \mathbb{Z}
\]
Even more, up to sign there is only one natural boundary map.

**Theorem 1.78.** Let \( \partial: K_{*+1}(Q) \rightarrow K_*(I) \) be a natural boundary map. Then there is \( \varepsilon \in \{\pm 1\} \) such that for all extensions \( \varepsilon \cdot \partial \) is the composition
\[
K_{*+1}(Q) \simeq KK_{*+1}(C, Q) \rightarrow KK_*(C, I) \simeq K_*(I)
\]
where the middle map is the Kasparov product with the class of the extension in \( KK_1(Q, I) \).
The same holds in K-homology.

### 1.7 Mayer-Vietoris sequences

Consider the category of pullback diagrams
\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & B
\end{array}
\]
A natural Mayer-Vietoris sequence is a functor from this category to the category of exact chain complexes, whose entries are \( K_*(A), K_*(A') \oplus K_*(B), K_*(B') \).

**Theorem 1.79.** Let \( d: K_*(B') \rightarrow K_{*+1}(A) \) be a boundary map in a natural Mayer-Vietoris sequence. Then there is a sign \( \varepsilon_\ast \in \{\pm \} \) such that for any pullback diagram \( \varepsilon \cdot d \) is the composition
\[
K_*(B') \xrightarrow{\delta} K_*(\ker(A' \rightarrow B'))
\]
\[
K_*(A) \leftarrow K_*(\ker(A \rightarrow B))
\]

Remark 1.80. To fix sign, one can look at pullback
\[
\begin{array}{ccc}
C_0((0, 1)) & \rightarrow & 0 \\
\downarrow & & \downarrow \\
C_0((0, 1]) & \rightarrow & C
\end{array}
\]
or its suspension.
Let $F$ be a homological functor on separable C*-algebras, and let $d : F_1(B') \to F_0(A)$ be a natural transformation on pullback diagrams

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
A' & \to & B'
\end{array}
\]

We compare a given structure to simpler one

\[
\begin{array}{ccc}
\ker p' & \to & 0 \\
\downarrow & & \downarrow \\
A' & \xrightarrow{\rho'} & B' \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}
\]

\[
\begin{array}{ccc}
F(B') & \xrightarrow{d_2} & F(\ker p') \\
\downarrow & & \downarrow \\
F(B') & \xrightarrow{d_1} & F(?)
\end{array}
\]

As a consequence, a natural transformation for pullback diagrams reduces to a natural transformation $E_1(Q) \to F_0(I)$ for extensions

\[
\begin{array}{ccc}
l & \xrightarrow{i} & E \\
\downarrow & & \downarrow \\
C_p & \xrightarrow{Z_p} & Q
\end{array}
\]

Next we compare this extension with mapping cylinder extension

\[
\begin{array}{ccc}
l & \xrightarrow{i} & E \\
\downarrow & & \downarrow \\
C_p & \xrightarrow{Z_p} & Q
\end{array}
\]

where

\[
Z_p := \{(e, q) \in E \oplus C([0, 1], Q) \mid p(e) = q(1)\}
\]

Now there are

\[
\begin{array}{ccc}
F_1(Q) & \xrightarrow{d_3} & F_0(I) \\
\downarrow & & \downarrow \\
F_1(Q) & \xrightarrow{d_4} & F_0(C_p)
\end{array}
\]

\[d_3 = F_0(\text{can})^{-1} \circ d_4 \]

If $p$ has a completely positive contractive section, then $F_0(I) \xrightarrow{\simeq} F_0(C_p)$. Actually if $F$ is exact, this is true without completely positive contractive section. Then the class of the extension in $KK_1(Q, I)$ is the product of

\[C_0((0, 1)) \otimes Q \hookrightarrow C_p \xrightarrow{\simeq} I\]

The map $I \hookrightarrow C_p$ has to be an $E$-equivalence because it is part of an extension

\[
\begin{array}{ccc}
l & \xrightarrow{i} & C_p \\
\downarrow & & \downarrow \\
C_0((0, 1], Q)
\end{array}
\]

and $C_0((0, 1], Q)$ is contractible.
Next we consider

\[
\begin{array}{cccccc}
C_p & \rightarrow & Z_p & \rightarrow & Q \\
SQ & \rightarrow & C_0([0, 1], Q) & \rightarrow & Q
\end{array}
\]

and

\[
\begin{array}{cccccc}
F_1(Q) & \xrightarrow{d_4} & F_0(C_p) & \quad d_5 = F_0(\text{can})^{-1} \circ d_4 \\
\Rightarrow & & \Rightarrow & & \Rightarrow \\
F_1(Q) & \xrightarrow{d_5} & F_0(SQ)
\end{array}
\]

This is \(d_5\) composed with the class of the extension \(I \hookrightarrow E \rightarrow Q\) in \(KK_0(SQ, I)\) or rather \(E_0(SQ, I)\) if there is no completely positive contractive section.

Now assume \(F_* = K_*\). We want to get rid of \(Q\). Now the boundary map for the cone extension of \(Q\) is a natural transformation \(K_1(Q) \rightarrow K_0(SQ)\). We have naturality of \(*\)-homomorphisms to begin with, but this implies naturality of \(KK_0\)-morphisms. Any \(x \in K_1(Q)\) is of the form \(\tilde{x}_*(g)\), where \(g \in K_1(C_0(\mathbb{R}))\) is the canonical generator, and \(\tilde{x} \in KK_0(C_0(\mathbb{R}), Q)\).

\[
K_1(Q) \simeq KK_0(C_0(\mathbb{R}), Q) \\
x \mapsto \tilde{x}
\]

\[
\begin{array}{cccccc}
x & \mapsto & K_1(Q) & \xrightarrow{d} & K_0(SQ) \\
\downarrow & & \downarrow \tilde{x} & & \downarrow \tilde{z} \\
g & \mapsto & K_1(C_0(\mathbb{R})) & \xrightarrow{d} & K_0(SC_0(\mathbb{R}))
\end{array}
\]

We conclude that \(d(x) = (\tilde{x})_*(d(g))\), so \(d\) is fixed completely once we know \(d(g) \in K_0(C_0(\mathbb{R}^2)) = \mathbb{Z}\). If we use an exact sequence

\[
\begin{array}{cccccc}
K_1(C_0([0, 1]), C_0(\mathbb{R})) & \rightarrow & K_1(C_0(\mathbb{R})) & \xrightarrow{\sim} & K_0(C_0(\mathbb{R}^2)) & \rightarrow & 0 \\
\rightarrow \mathbb{Z} & & \Rightarrow & & \Rightarrow & & \Rightarrow
\end{array}
\]

we conclude that \(d(g)\) has to be a generator of \(K_0(C_0(\mathbb{R}^2)) \simeq \mathbb{Z}\), and

\[
K_1(Q) \simeq KK_0(C_0(\mathbb{R}), Q) \simeq KK_0(\mathbb{C}, C_0(\mathbb{R}, Q)) \simeq K_0(C_0(\mathbb{R}, Q))
\]

\[
x \mapsto \tilde{x}
\]

We fix natural isomorphisms

\[
K_1(Q) \simeq KK_0(C_0(\mathbb{R}), Q) \simeq KK_0(\mathbb{C}, C_0(\mathbb{R}) \otimes Q) \simeq K_1(SQ)
\]

which are unique up to sign. Then \(d\) is this isomorphism up to sign.

For the boundary map \(K_0(Q) \rightarrow K_1(SQ)\) the same thing happens, but replacing \(g\) by the generator of \(K_0(\mathbb{C})\).

Let \(x \in K_1(Q)\), \(y \in K^0(I)\).

\[
\mathbb{C} \rightarrow Q \xrightarrow{[E]} I
\]

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Using Kasparov product $\circ$ we write
\[
\begin{align*}
\partial(x) &= \varepsilon_{\partial}[E] \circ x \\
\delta(y) &= \varepsilon_{\delta y}[E] \\
\langle x, \delta y \rangle &= \delta(y) \circ x = \varepsilon_{\delta}(y) \circ [E] \circ x \in KK_0(C, C) \cong \mathbb{Z} \\
\langle \partial(x), y \rangle &= y \circ \partial(x) = \varepsilon_{\partial}(y) \circ [E] \circ x
\end{align*}
\]

1.8 Localisation of functors

Assume we have a triangulated category $\mathcal{T}$ with $\oplus$, a localising subcategory $\mathcal{N}$ and a class of objects $\mathcal{P}$ such that $(\langle \mathcal{P} \rangle, \mathcal{N})$ is complementary. For example we can take $\mathcal{T} = KK$, $\mathcal{N} = \{ \mathcal{B} \in KK \mid K_*\mathcal{B} = 0 \}$, $\mathcal{P} = \{ \mathcal{C} \}$. Furthermore, let $F : \mathcal{T} \to \mathcal{A}$ be a homological functor commuting with $\oplus$. Recall that there are functors
\[
P : \mathcal{T} \to \langle \mathcal{P} \rangle, \quad N : \mathcal{T} \to \mathcal{N}
\]
and natural exact triangles
\[
P(B) \to B \to N(B) \to P(B)[1]
\]

**Definition 1.81.** The localisation of functor $F$ at $\mathcal{N}$, denoted $LF$, is a functor
\[
F \circ P : \mathcal{T} \to \mathcal{A}
\]

We may also view this as a functor on $\mathcal{T}/\mathcal{N}$. There is a natural transformation $LF \to F$.

**Proposition 1.82.** $LF \to F$ is universal among natural transformations $G \to F$ with $G$ homological and $G/\mathcal{N} = 0$

\[
\begin{array}{ccc}
G & \longrightarrow & F \\
\downarrow & & \downarrow \\
& \searrow & \nearrow \\
& LF & \\
\end{array}
\]

**Proof.** There is an isomorphism
\[
G(P(B)) \cong G(B)
\]
and a map
\[
G(P(B)) \to F(P(B)) = LF(B)
\]

Roughly speaking, $LF$ is the best approximation to $F$ that vanishes on $\mathcal{N}$.

**Corollary 1.83.** If $LF \to F$ is invertible, then $F|_{\mathcal{N}} = 0$.

**Proposition 1.84.** Let $G, F$ be homological, commuting with $\oplus$, $G/\mathcal{N} = 0$, and let $\Phi : G \to F$ be a natural transformation. Then if $\Phi_B$ is invertible for all $B \in \mathcal{P}$, then $\Phi$ induces a natural isomorphism $G \cong LF$.

**Proof.** We get a transformation $\Psi : G \to LF$ by the previous proposition. $\Psi$ is invertible on $\mathcal{P}$ because $LF(B) \cong F(B)$ for $B \in \mathcal{P}$. Since $G$ and $LF$ are homological and commuting with $\oplus$, the class of objects where $\Psi$ is invertible is localising. Hence contains $\mathcal{P}$. It also contains $\mathcal{N}$ because $G$ and $LF$ vanish on $\mathcal{N}$. Thus it contains $\mathcal{T}$. 

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Usually we do not expect the map $L F \to F$ to be an isomorphism. But sometimes in noncommutative topology this happens for rather deep reason. For example the Baum-Connes assembly map is of this form for suitable choice of $\mathcal{N}$ and $F(B) = K_*(G \rtimes_r B)$.

Let $T = KK^G$, $G$ locally compact group. How to chose $\mathcal{N}$? In the group case the following choice is most useful

$$B \in \mathcal{N} \text{ if and only if } \text{Res}^H_G(B) \simeq 0 \text{ in } \text{KK}^H, \text{ for all compact subgroups } H \leq G.$$  

This definition contains the insight that the K-theory for crossed products by compact groups has to be computes by hand, whereas those for non-compact groups often reduce to compact groups.

**Theorem 1.85.** Let $T = KK^G$ for a Lie group $G$, and $F(B) = K_*(G \rtimes_r B)$, $\mathcal{N}$ as above. Then the natural transformation $L F \to F$ is naturally isomorphic to the Baum-Connes assembly map with coeffictients.

**Proof.** The domain of the Baum-Connes map

$$K_*^{\text{top}}(G, B) = \lim_{X \in \mathcal{E} G, X G-\text{compact}} KK^G(C_0(X), B)$$

has two properties

- it vanishes for $B \in \mathcal{N}$

$$KK^G(C_0(X), B) \to KK(G \rtimes_r C_0(X), G \rtimes_r B) \to K_*(G \rtimes_r B)$$

- the Baum-Connes assembly map is invertible

\[\Box\]

**Definition 1.86.** A $G$-algebra is called proper Husdorff if there is a proper $G$-space $X$ and a continuous $G$-map $\text{Prim}(A) \to X$ (equivalently $C_0(X) \to A$ is central).

### 1.9 Towards an analogue of the Baum-connes conjecture for quantum groups

The main question is: what are good choices for $\mathcal{P}$, $\mathcal{N}$? We must choose $\mathcal{N}$, $\mathcal{P}$ so that the resulting assembly map is invertible for ”nice” quantum groups. first approach is to use restriction functors to all compact quantum subgroups.

**Definition 1.87.** A quantum group is a $C^*$-algebra $A$ with a comultiplication $\Delta: A \to A \otimes A$ satisfying certain properties.

A quantum group is compact if $A$ is unital.

**Example 1.88.** Right now, we had only two examples: groups and their duals

1. $A = C_0(G)$
   $$\Delta: C_0(G) \to C_0(G \times G), \quad (\Delta f)(x, y) = f(xy).$$

2. $A = C^*_r(G)$,
   $$\Delta: C^*_r(G) \to M(C^*_r(G) \otimes C^*_r(G)), \quad \Delta \left( \int_G f(t) \lambda_t dt \right) = \int_G f(t) \lambda_t \otimes \lambda_t dt.$$
Group actions on C*-algebras become coactions of \((A, \Delta)\)
\[
\delta_B : B \rightarrow M(B \otimes A)
\]
coadsociative plus technical conditions.

**Example 1.89.**
1. Group actions as usual.
2. Grading by \(G\).

**Definition 1.90.** A closed quantum subgroup of \((A, \Delta)\) is a quotient \(A/I\) to which \(\Delta\) descends.

**Example 1.91.**
1. Closed quantum subgroups of \(C_0(G)\) are \(C_0(H)\) for \(H \leq G\) closed subgroup.
2. Closed quantum subgroups of \(C^*_r(G)\) are too few. The candidates are \(C^*_r(G/N)\), where \(N \leq G\) is a closed normal subgroup.

Many locally compact groups such as \(\text{GL}_2(\mathbb{Q}_p)\) have many open subgroups but no open normal subgroup.

**Definition 1.92.** A quantum homogeneous space for \((A, \Delta)\) is a C*-subalgebra \(B\) of \(M(A)\) that is a left \(\Delta\)-coideal \((\Delta(B) \subseteq M(B \otimes A))\). It is proper if \(B \not\subseteq A\).

**Example 1.93.**
1. \(B = C_0(G/H), H \subseteq G\) closed subgroup.
2. \(C^*_r(H)\), for any closed subgroup of \(H \subseteq G\) is even a two-sided Cl-coideal. Proper homogeneous spaces are open subgroups here.

Let us look at \(C^*_r(G)\) when \(G\) is a compact Lie group. Then the following conditions are equivalent
1. \(G\) is connected.
2. \(G\) has no open subgroups.
3. \(C^*_r(G)\) has no non-trivial proper homogeneous spaces.

But \(G = \text{SO}(3)\) creates a problem because it has projective representations. \(G\) acts on \(M_2(\mathbb{C})\) because of the representation of \(\text{SO}(3)\) on \(\mathbb{C}^2\). \(G\) coacts on \(G \rtimes_r M_2(\mathbb{C})\).

What are particularly simple actions of a quantum group?
\[
C_0(G/H) \rtimes_r G \sim_{M.E.} C^*(H) \simeq \bigoplus_{\pi \in \hat{G}} M_{d_\pi}(\mathbb{C})
\]

A necessary condition for a torsion coefficient algebra is that the crossed product \(B \rtimes_r A\) be a sum of matrix algebras (compact operators).

**Theorem 1.94.** Let \(G\) be a locally compact group.
\[
P := \{C_0(G/H) \mid H \leq G, \text{ compact}\}
\]
\[
\tilde{\mathcal{N}} = \mathcal{P}^+ := \{B \mid \text{KK}^G(P, B) = 0 \text{ for all } P \in \mathcal{P}\}
\]
The localisation of \(K_*(G \rtimes B)\) at \(\tilde{\mathcal{N}}\) and \(\mathcal{N}\) agree with the domain of the Baum-Connes assembly map
\[
\mathcal{N} = \{B \mid \text{Res}^H_B \simeq 0 \text{ for all compact } H \leq G\}.
\]
1.10 Quantum groups

Definition 1.95. A quantum group is a C*-algebra $A$ with a comultiplication $\Delta \in \text{Mor}(A, A \otimes A)$ such that

$$\Delta(A)(1 \otimes A) = A \otimes A$$

$$\Delta : A \rightarrow M(A \otimes A)$$

$$\otimes = \otimes_{\text{min}}, \quad 1_A \in M(A)$$

and for all $a, b \in A$

$$\Delta(a)(1 \otimes b) \in A \otimes A$$

$$(a \otimes 1)\Delta(b) \in A \otimes A$$

$\text{span}\{\Delta(a)(1 \otimes b) \mid a, b \in A\}$ is dense in $A \otimes A$

$\text{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\}$ is dense in $A \otimes A$

in the compact case, that is when $1_A \in A$ we have

$$(\Delta \circ \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$$

$A \xrightarrow{\Delta} M(A \otimes A) \xrightarrow{\Delta \otimes \text{id}} M(A \otimes A \otimes A)$

Theorem 1.96. There is a unique state $h$ on $A$ such that

$$(\text{id} \otimes h)\Delta(A) = h(a)1_A = (h \otimes \text{id})\Delta(a)$$

Let $G$ be a locally compact quantum group, $A = C_0(G)$, $(\Delta f)(x, y) = f(xy)$. Here $\Delta \in \text{Mor}(A, A \otimes A)$ is induced by the group multiplication $\mu : G \times G \rightarrow G$. Multiplication $\mu$ is associative if $\Delta$ is coassociative. The conditions

$\text{span}\{\Delta(a)(1 \otimes b) \mid a, b \in A\}$ is dense in $A \otimes A$

$\text{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\}$ is dense in $A \otimes A$

can be written as

$$\exists x \quad \mu(xy) = \mu(xz) \implies y = z$$

$$\exists x \quad \mu(yx) = \mu(zx) \implies y = z$$

On a group Haar measure $h$ satisfies

$$\int_G f(st)dh(s) = \int_G f(s)dh(s)$$

Definition 1.97. A function $h : A_+ \rightarrow [0, \infty]$ such that $h(a+b) = h(a) + h(b)$, $h(\lambda a) = \lambda h(a)$ for $\lambda \geq 0$ is called a weight.
We define

\[ \mathcal{N}_h := \{ a \mid h(a^*a) < \infty \} \quad (L^2) \]
\[ \mathcal{M}_h := \text{span}\{ a \geq 0 \mid h(a) < \infty \} \]
\[ = \text{span}\{ a^*b \mid a, b \in \mathcal{N}_h \} \]

Then \( \overline{\mathcal{M}_h} = A \) (\( h \) locally finite), and \((\text{id} \otimes h)\Delta(a) = h(a)1_A\) (\( h \) lower semicontinuous).

Let \( \varphi \in A^* \), \( a \in A \). Then

\[ \varphi * a := (\text{id} \otimes \varphi)\Delta(a). \]

In particular, for \( \varphi = \delta_t \)

\[ (\varphi * a)(s) = a(st). \]

Right invariance of \( h \) means that

\[ h(\varphi * a) = h(a)\varphi(1_A) \]

for all \( \varphi \in A^* \), and all \( a \geq 0 \).

We say that \( h \) is strictly faithful if

\[ h(a^*a) = 0 \implies a = 0. \]

There exists \( \kappa \) - closed densely defined map \( A \to A \), such that

\[ \kappa = R \circ \tau_{i/2}, \]

where \( R \) is an antiautomorphism, and \( \tau_{i/2} \) is an analytic extension of a 1-parameter group \((\tau_t)_{t \in \mathbb{R}}\) of automorphisms of \( A \). There exists \( \lambda > 0 \) such that \( h \circ \tau_t = \lambda^t h \).

For all \( \varphi \in A^* \), \( \varphi \circ \kappa \in A \) and all \( a, b \in \mathcal{N}_h \)

\[ h((\varphi * a^*)b) = h(a^*((\varphi \circ \kappa) * b)) \]

Strong right invariance means that

\[ \mu(\kappa \otimes \text{id})\Delta(a) = \varepsilon(1)1_A = \mu(\text{id} \otimes \kappa)\Delta(a) \]

The maps

\[ \Phi: a \otimes b \mapsto \Delta(a)(1_A \otimes b) \]
\[ \Psi: r \otimes s \mapsto (\text{id} \otimes \kappa)(\Delta(r))(1 \otimes s) \]

are inverse to each other.

We can embed \( A \) in a Hilbert space \( \mathcal{H} \) and extend \( \Phi, \Psi \) to

\[ W: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \]
\[ V: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \]

Strong right invariance means that \( W^* = V \)

\[ \langle W(a \otimes b), c \otimes d \rangle = \langle a \otimes b, V(c \otimes d) \rangle. \]
1.11 The Baum-Connes conjecture

Let $G$ be a torsion-free group, that is without compact subgroups. The Baum-Connes conjecture with coefficients for $G$ means that $K_\ast(G \rtimes_r A) = 0$ whenever $K_\ast(A) = 0$. If $G$ has torsion, then the statement is: if $K_\ast(A \rtimes_r H) = 0$ for all $H \leq G$ compact, then $K_\ast(A \rtimes_r G) = 0$.

**Theorem 1.98** (Higson-Kasparov). The Baum-Connes conjecture with coefficients holds for all amenable groups.

In particular it holds if $G = \mathbb{Z}^n$ for some $n \in \mathbb{N}$.

Let

\[ \mathcal{N} := \{ A \in KK^G \mid K_\ast(A \rtimes H) = 0 \text{ for all compact } H \leq G \} \]
\[ \mathcal{N}^+ := \{ A \in KK^G \mid KK^G(A, B) = 0 \text{ for all } B \in \mathcal{N} \} \]

for a discrete $G$. Then $(\mathcal{N}^+, \mathcal{N})$ are complementary.