Equivariant KK-theory and noncommutative index theory

Paul F. Baum

Notes by
Pawel Witkowski

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Chapter 1

KK-theory

1.1 C*-algebras

Let $G$ be a locally compact, Hausdorff, second countable (the topology of $G$ has a countable base) group. Examples are:

- Lie groups with $\pi_0(G)$ finite - $\text{SL}(n, \mathbb{R})$,
- $p$-adic groups - $\text{SL}(n, \mathbb{Q}_p)$,
- adelic groups - $\text{SL}(n, A)$,
- discrete groups - $\text{SL}(n, \mathbb{Z})$.

For a group $G$ we have the reduced C*-algebra of $G$, denoted by $C^*_{r}G$. The problem is to compute its K-theory $K_j(C^*_{r}G)$, $j = 0, 1$.

**Conjecture 1 (P. Baum - A. Connes).** For all locally compact, Hausdorff, second countable groups $G$

$$\mu: K_j^G(EG) \to K_j(C^*_{r}G)$$

is an isomorphism for $j = 0, 1$.

Recall some definitions:

**Definition 1.1.** A Banach algebra is an algebra $A$ over $\mathbb{C}$ with a given norm $\| \cdot \|$ such that $A$ is complete normed algebra, i.e.

- $\|\lambda a\| = |\lambda|\|a\|$, $\lambda \in \mathbb{C}$, $a \in A$,
- $\|a + b\| \leq \|a\| + \|b\|$, $a, b \in A$,
- $\|ab\| \leq \|a\|\|b\|$, $a, b \in A$,
- $\|a\| = 0$ if and only if $a = 0$,

and every Cauchy sequence is convergent in $A$ (with respect to the metric $\|a - b\|$).

**Definition 1.2.** A C*-algebra is a Banach algebra $(A, \| \cdot \|)$ with a map $*: A \to A$, $a \mapsto a^*$ satisfying

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\begin{itemize}
\item \((a^*)^* = a,\)
\item \((a + b)^* = a^* + b^*,\)
\item \((ab)^* = b^* a^*,\)
\item \((\lambda a)^* = \lambda a^*, \ a, b \in A, \lambda \in \mathbb{C},\)
\item \(\|aa^*\| = \|a\|^2 = \|a^*\|^2.\)
\end{itemize}

A \textbf{*-morphism} is an algebra homomorphism \(\varphi: A \to B\) such that \(\varphi(a^*) = (\varphi(a))^*\) for all \(a \in A.\)

\textbf{Lemma 1.3.} If \(\varphi: A \to B\) is a \(\textbf{*}-\text{homomorphism}\) then \(\|\varphi(a)\| \leq \|a\|\) for all \(a \in A.\)

\textbf{Example 1.4.} Let \(X\) be a locally compact Hausdorff topological space, and \(X^+ = X \cup \{p_\infty\}\) its one-point compactification. Define
\[
C_0(X) := \{\alpha: X^+ \to \mathbb{C} \mid \alpha \text{ is continuous}, \alpha(p_\infty) = 0\},
\]
\[
\|\alpha\| = \sup_{p \in X} |\alpha(p)|, \quad \alpha^*(p) = \overline{\alpha(p)}.
\]

with operations
\[
(\alpha + \beta)(p) = \alpha(p) + \beta(p),
\]
\[
(\alpha\beta)(p) = \alpha(p)\beta(p),
\]
\[
(\lambda \alpha)(p) = \lambda \alpha(p), \ \lambda \in \mathbb{C}.
\]

If \(X\) is compact, then
\[
C_0(X) := C(X) = \{\alpha: X \to \mathbb{C} \mid \alpha \text{ is continuous}\},
\]

\textbf{Example 1.5.} Let \(\mathcal{H}\) be a separable Hilbert space (admits a countable or finite orthonormal basis). Define
\[
\mathcal{L}(\mathcal{H}) := \{T: \mathcal{H} \to \mathcal{H} \mid T \text{ bounded}\},
\]
\[
\|T\| = \sup_{u \in \mathcal{H}, \|u\| = 1} \|Tu\|, \quad \|u\| = \sqrt{\langle u, u \rangle},
\]
\[
\langle Tu, v \rangle = \langle u, T^*v \rangle \text{ for all } u, v \in \mathcal{H}.
\]

with operations
\[
(T + S)u = Tu + Su,
\]
\[
(TS)u = T(Su),
\]
\[
(\lambda T)u = \lambda (Tu), \ \lambda \in \mathbb{C}.
\]

\textbf{Example 1.6.} If \(\mathcal{H}\) is a Hilbert space, then define
\[
\mathcal{K}(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \mid T \text{ is compact operator}\}
\]
\[
= \{T \in \mathcal{L}(\mathcal{H}) \mid \dim_{\mathbb{C}} T(\mathcal{H}) < \infty\}
\]

with the closure in operator norm. Then \(\mathcal{K}(\mathcal{H})\) is a sub-C*-algebra of \(\mathcal{L}(\mathcal{H})\) and an ideal in \(\mathcal{L}(\mathcal{H}).\)
Example 1.7. Let $G$ be a locally compact Hausdorff second countable topological group. Fix a left-invariant Haar measure $dg$ for $G$, that is for all continuous $f: G \to \mathbb{C}$ with compact support

$$\int_G f(\gamma g) dg = \int_G f(g) dg$$

for all $\gamma \in G$.

Let $L^2G$ be the following Hilbert space

$$L^2G = \{ u: G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty \}$$

$$\langle u, v \rangle = \int_G \overline{u(g)}v(g) dg, \quad u, v \in L^2G.$$ 

Let $\mathcal{L}(L^2G)$ be the C*-algebra of all bounded operators $T: L^2G \to L^2G$. Let

$$C_cG = \{ f: G \to \mathbb{C} \mid f \text{ is continuous, and has compact support} \}.$$ 

Then $C_cG$ is an algebra

$$(\lambda f)g = \lambda (fg), \quad \lambda \in \mathbb{C}, \quad g \in G$$

$$(f + h)g = fg + hg$$

$$(f \ast h)g_0 = \int_G f(g)h(g^{-1}g_0) dg, \quad g_0 \in G.$$ 

There is an injection of algebras

$$0 \to C_cG \to \mathcal{L}(L^2G)$$

given by $f \mapsto T_f$, $T_f(u) = f \ast u$, $u \in L^2G$,

$$(f \ast u)g_0 = \int_G f(g)u(g^{-1}g_0) dg, \quad g_0 \in G.$$ 

Define the reduced C*-algebra $C^*_rG$ of $G$ as the closure of $C_cG \subset \mathcal{L}(L^2G)$ in the operator norm. $C^*_rG$ is a sub-C*-algebra of $\mathcal{L}(L^2G)$.

Definition 1.8. A subalgebra $A$ of $\mathcal{L}(\mathcal{H})$ is a C*-algebra of operators if and only if

1. $A$ is closed with respect to the operator norm.

2. If $T \in A$, then the adjoint operator $T^* \in A$.

Theorem 1.9 (I. Gelfand, V. Naimark). Any C*-algebra is isomorphic, as a C*-algebra, to a C*-algebra of operators.

Theorem 1.10. Let $A$ be a commutative C*-algebra. Then $A$ is (canonically) isomorphic to $C_0(X)$ where $X$ is the space of maximal ideals of $A$.

Thus a non-commutative C*-algebra can be viewed as a ”noncommutative locally compact Hausdorff topological space”.

We have an equivalence of the following categories

- Commutative C*-algebras with *-homomorphisms,
- Locally compact Hausdorff topological spaces with morphisms from $X$ to $Y$ being a continuous maps $f: X^+ \to Y^+$ with $f(p_\infty) = q_\infty$. 

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1.2 K-theory

Let $A$ be a C*-algebra with unit $1_A$,
\[ K_0(A) = K_0^{alg}(A) = \text{Grothendieck group of finitely generated} \]
(\text{left) projective} \ A\text{-modules}

In the definition of $K_0(A)$ we can forget about $\| \cdot \|$ and $\ast$. In the definition of $K_1(A)$ we cannot forget about that.

Take a topological groups $\text{GL}(n,A)$ and embeddings $\text{GL}(n,A) \hookrightarrow \text{GL}(n+1,A)$
\[
\begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nn}
\end{pmatrix} \mapsto
\begin{pmatrix}
  a_{11} & \cdots & a_{1n} & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  a_{n1} & \cdots & a_{nn} & 0 \\
  0 & \cdots & 0 & 1_A
\end{pmatrix}
\]

Then $\text{GL}(A) = \lim_{n \to \infty} \text{GL}(n,A)$ with the direct limit topology. Define the K-theory groups
\[ K_j(A) := \pi_{j-1}(\text{GL}(A)), \quad j = 1, 2, 3, \ldots \]

Bott periodicity states that $\Omega^2 \text{GL}(A) \sim \text{GL}(A)$, so $K_j(A) \simeq K_{j+2}(A)$ for $j = 0, 1, 2, \ldots$. Thus in fact we have two groups $K_0(A)$ and $K_1(A)$.

If $A$ is not unital, then we can adjoin a unit,
\[ 0 \to A \to \tilde{A} \to \mathbb{C} \to 0 \]

and define
\[ K_0(A) := \ker(K_0(\tilde{A}) \to K_0(\mathbb{C})), \]
\[ K_1(A) := K_1(\tilde{A}). \]

If $\varphi: A \to B$ is a *-homomorphism, then there is an induced homomorphism of abelian groups $K_j(A) \to K_j(B)$.

**Example 1.11.** $\mathbb{C}$ is a C*-algebra, $\| \lambda \| = |\lambda|$, $\lambda^* = \bar{\lambda}$.

**Theorem 1.12 (Bott).**
\[ K_j(\mathbb{C}) = \begin{cases} 
\mathbb{Z} & j \text{ even} \\
0 & j \text{ odd} 
\end{cases} \]

**Theorem 1.13 (Bott).**
\[ \pi_j(\text{GL}(n,\mathbb{C})) = \begin{cases} 
0 & j \text{ even} \\
\mathbb{Z} & j \text{ odd} 
\end{cases} \]

for $j = 0, 1, \ldots, 2n - 1$.

For a locally compact Hausdorff topological space one defines a topological K-theory with compact supports (Atiyah-Hirzebruch)
\[ K^j(X) := K_j(C_0(X)). \]

If $X$ is compact Hausdorff then $K^0(X)$ is the Grothendieck group of complex vector bundles on $X$.

There is a chern character
\[ \text{ch}: K^j(X) \to \bigoplus_l H^{j+2l}_c(X;\mathbb{Q}), \quad j = 0, 1. \]
Theorem 1.14. For any locally compact Hausdorff topological space $X$

$$\text{ch} : K^j(X) \to \bigoplus_l H^{j+2l}_c(X; \mathbb{Q})$$

is a rational isomorphism, i.e.

$$\text{ch} : K^j(X) \otimes \mathbb{Z} \mathbb{Q} \to \bigoplus_l H^{j+2l}_c(X; \mathbb{Q})$$

is an isomorphism for $j = 0, 1$.

We can use Čech cohomology, Alexander-Spanier cohomology or representable cohomology (all with compact supports).

1.3 Representations

Definition 1.15. A representation of C*-algebra $A$ is a *-homomorphism

$$\varphi : A \to \mathcal{L}(\mathcal{H})$$

where $\mathcal{H}$ is a Hilbert space.

The myth: for a reduced C*-algebra $C^*_r G$ of $G$ there exists a locally compact Hausdorff topological space $\hat{G}_r$. The space $\hat{G}_r$ has one point for each distinct (i.e. non-equivalent) irreducible unitary representation of $G$ which is weakly contained in the (left) regular representation of $G$. $\hat{G}_r$ is known as the support of the Plancherel measure or the reduced unitary dual of $G$. The K-theory $K_*(C^*_r G)$ is the topological K-theory (with compact supports of $\hat{G}_r$).

Example 1.16. For $G = \text{SL}(2, \mathbb{R})$ we have $\hat{G}_r$:

```
  ...
  ..
  ..
  ..
  ..
  ..
  ..
  ..
```

1.4 K-homology

Let $A$ be a separable C*-algebra ($A$ has a countable dense subset). We will define generalized elliptic operators over $A$ in the odd and even case.

Definition 1.17 (odd case). A generalized odd elliptic operator over $A$ is a triple $(\mathcal{H}, \psi, T)$ such that

1. $\mathcal{H}$ is a separable Hilbert space,

2. $\psi : A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphisms,

3. $T \in \mathcal{L}(\mathcal{H})$
and
\[ T = T^*, \quad \psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - T^2) \in \mathcal{K}(\mathcal{H}) \]
for all \( a \in A \).

We will denote the set of such triples by \( \mathcal{E}^1(A) \). If \( \varphi : A \to B \) is a \(^*\)-homomorphism then there is an induced map
\[ \varphi^* : \mathcal{E}^1(B) \to \mathcal{E}^1(A), \quad \varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T). \]

**Example 1.18.** \( S^1 := \{(t_1, t_2) \in \mathbb{R} \mid t_1^2 + t_2^2 = 1\}, \quad A = C(S^1), \quad \psi : C(S^1) \to \mathcal{L}(L^2(S^1)) \)
\[ \psi(a)(u) = \alpha(u), \quad \alpha \in C(S^1), \quad u \in L^2(S^1), \]
\[ (\alpha u)(\lambda) = \alpha(\lambda)u(\lambda), \quad \lambda \in S^1. \]
The Dirac operator \( D \) of \( S^1 \) is \(-i\frac{\partial}{\partial \theta}\). If we take a basis \( \{e^{in\theta}\}_{n \in \mathbb{Z}} \) of \( L^2(S^1) \), then
\[ D(e^{in\theta}) = \left(-i\frac{\partial}{\partial \theta}\right)(e^{in\theta}) = ne^{in\theta}. \]

Set \( T = D(I + DD)^{-\frac{1}{2}}. \) Then
\[ T(e^{in\theta}) = \frac{n}{\sqrt{1 + n^2}}e^{in\theta}, \]
and \( (L^2(S^1), \psi, T) \in \mathcal{E}^1(C(S^1)) \).

We will define odd K-homology of \( A \) by
\[ K^1(A) := \mathcal{E}^1(A)/\sim (= \text{KK}(A, \mathbb{C})), \]
where the relation \( \sim \) is homotopy, which is defined below.

**Definition 1.19.** Let \( \xi = (\mathcal{H}, \psi, T), \eta = (\mathcal{H}', \psi', T') \) be elements of \( \mathcal{E}^1(A) \). We say that \( \xi \) is isomorphic to \( \eta \), \( \xi \simeq \eta \) if there exists a unitary operator \( U : \mathcal{H} \to \mathcal{H}' \) with commutativity in the diagrams
\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{U} & \mathcal{H}' \\
\downarrow T & & \downarrow \psi(a) \\
\mathcal{H} & \xrightarrow{U} & \mathcal{H}'
\end{array}
\quad
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{U} & \mathcal{H}' \\
\downarrow T & & \downarrow \psi'(a) \\
\mathcal{H} & \xrightarrow{U} & \mathcal{H}'
\end{array}
\]
for all \( a \in A \).

**Definition 1.20.** We say that \( \xi = (\mathcal{H}, \psi, T), \eta = (\mathcal{H}', \psi', T') \in \mathcal{E}^1(A) \) are strictly homotopic if there exists a continuous function \([0, 1] \to \mathcal{L}(\mathcal{H})\), \( t \mapsto T_t \) such that
1. \( T_0 = T \),
2. for all \( t \in [0, 1], \quad (\mathcal{H}, \psi, T_t) \in \mathcal{E}^1(A), \)
3. \( (\mathcal{H}, \psi, T_1) \simeq (\mathcal{H}', \psi', T'). \)

**Definition 1.21.** We say that a generalized elliptic operator \((\mathcal{H}, \psi, T) \in \mathcal{E}^1(A)\) is degenerate if and only if
\[ \psi(a)T - T\psi(a) = 0, \quad \psi(a)(I - T^2) = 0, \quad \text{for all} \ a \in A. \]

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Definition 1.22. We say that \( \xi = (H, \psi, T) \), \( \eta = (H', \psi', T') \) \( \in E^1(A) \) are homotopic, \( \xi \sim \eta \), if and only if there exists degenerate generalized elliptic operators \( \hat{\xi}, \hat{\eta} \) with \( \xi \oplus \hat{\xi} \) strictly homotopic to \( \eta \oplus \hat{\eta} \).

Definition 1.23. Odd K-homology of a C*-algebra \( A \) is defined as the group of homotopy classes of generalized odd elliptic operators,

\[
K^1(A) \defeq E^1(A)/\sim
\]

It is an abelian group with respect to

\[
(H, \psi, T) + (H', \psi', T') = (H \oplus H', \psi \oplus \psi', T \oplus T')
\]

with inverse defined by

\[
-(H, \psi, T) = (H, \psi, -T).
\]

If \( \varphi : A \to B \) is a *-homomorphism, then there is an induced map

\[
\varphi^* : K^1(B) \to K^1(A), \quad \varphi^*(H, \psi, T) = (H, \psi \circ \varphi, T).
\]

Now we will define even elliptic operators and \( K^0(A) \).

Definition 1.24 (even case). A generalized even elliptic operator over \( A \) is a triple \((H, \psi, T)\) such that

1. \( H \) is a separable Hilbert space,
2. \( \psi : A \to \mathcal{L}(H) \) is a *-homomorphisms,
3. \( T \in \mathcal{L}(H) \)

and

\[
\psi(a)T - T\psi(a) \in \mathcal{K}(H), \quad \psi(a)(1 - TT^*) \in \mathcal{K}(H), \quad \psi(a)(1 - T^*T) \in \mathcal{K}(H)
\]

for all \( a \in A \).

We will denote the set of such triples by \( E^0(A) \).

Definition 1.25. Even K-homology of a C*-algebra \( A \) is defined as the group of homotopy classes of generalized even elliptic operators,

\[
K^0(A) \defeq E^0(A)/\sim
\]

It is an abelian group with respect to

\[
(H, \psi, T) + (H', \psi', T') = (H \oplus H', \psi \oplus \psi', T \oplus T')
\]

with inverse defined by

\[
-(H, \psi, T) = (H, \psi, -T).
\]

If \( \varphi : A \to B \) is a *-homomorphism, then there is an induced map

\[
\varphi^* : K^0(B) \to K^0(A), \quad \varphi^*(H, \psi, T) = (H, \psi \circ \varphi, T).
\]
1.5 Equivariant K-homology

Let $G$ be a locally compact Hausdorff second countable group, and $\mathcal{H}$ a separable Hilbert space. Denote the set of unitary operators on $\mathcal{H}$ by

$$\mathcal{U}(\mathcal{H}) := \{ U \in \mathcal{L}(\mathcal{H}) \mid UU^* = U^*U = I \}$$

**Definition 1.26.** A unitary representation of $G$ is a group homomorphism $\pi: G \to \mathcal{U}(\mathcal{H})$ such that for each $v \in \mathcal{H}$ the map $G \to \mathcal{H}$, $g \mapsto \pi(g)v$ is a continuous map from $G$ to $\mathcal{H}$.

**Definition 1.27.** A $G$-$C^*$-algebra is a $C^*$-algebra $A$ with a given continuous action $G \times A \to A$ by automorphisms.

**Example 1.28.** Let $X$ be a locally compact $G$-space. Then $G$ acts on $C_0(X)$ by

$$(ga)(x) = \alpha(g^{-1}x), \quad g \in G, \quad \alpha \in C_0(X), \quad x \in X.$$ This makes $C_0(X)$ a $G$-$C^*$-algebra.

Let $A$ be a (separable) $G$-$C^*$-algebra.

**Definition 1.29.** A covariant representation of $A$ is a triple $(H, \psi, \pi)$ such that

- $H$ is a separable Hilbert space,
- $\psi: A \to \mathcal{L}(H)$ is a *-homomorphism,
- $\pi: G \to \mathcal{U}(H)$ is a unitary representation of $G$,
- and

$$\psi(ga) = \pi(g)\psi(a)\pi(g^{-1})$$

for all $g \in G$, $a \in A$.

**Definition 1.30.** Equivariant odd K-homology $K^1_G(A)$ of a $G$-$C^*$-algebra $A$ is the group of homotopy classes of quadriples $(H, \psi, T, \pi)$, where $(H, \psi, \pi)$ is a covariant representation of $A$, and $T \in \mathcal{L}(H)$ is such that

$$T = T^*, \quad \pi(g)T - T\pi(g) \in \mathcal{K}(H), \quad \psi(a)T - T\psi(a) \in \mathcal{K}(H), \quad \psi(a)(1 - T^2) \in \mathcal{K}(H)$$

for all $g \in G$, $a \in A$.

$$K^1_G(A) = \{(H, \psi, \pi, T) \}/\sim$$

**Example 1.31.** Let $G = \mathbb{Z}$, $X = \mathbb{R}$, $A = C_0(\mathbb{R})$. Consider the action by translations

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}, \quad (n, t) \mapsto n + t.$$ Let $\mathcal{H} = L^2(\mathbb{R})$. Define $\psi: A \to \mathcal{L}(\mathcal{H})$ by

$$\psi(\alpha)u = \alpha u, \quad \alpha u(t) = \alpha(t)u(t), \quad \alpha \in C_0(\mathbb{R}), \quad u \in L^2(\mathbb{R}), \quad t \in \mathbb{R}.$$ The representation $\pi: \mathbb{Z} \to \mathcal{U}(L^2(\mathbb{R}))$ is defined by

$$(\pi(n)u)(t) := u(t - n).$$
As an operator on $L^2(\mathbb{R})$ we take $-i \frac{d}{dx}$. It is not a bounded operator on $L^2(\mathbb{R})$, but we can “normalize” it to obtain a bounded operator $T$. Since $-i \frac{d}{dx}$ is self-adjoint there is functional calculus, and $T$ can be taken to be the function $\frac{x}{\sqrt{1+x^2}}$ applied to $-i \frac{d}{dx}$,

$$T := \left( \frac{x}{\sqrt{1+x^2}} \right) (-i \frac{d}{dx}).$$

Equivalently, $T$ can be constructed using Fourier transform. Let $\mathcal{M}_x$ be the operator of “multiplication by $x$”

$$(\mathcal{M}_x f)(x) = xf(x).$$

Fourier transform converts $-i \frac{d}{dx}$ to $\mathcal{M}_x$, i.e. there is a commutativity in the diagram

$$\begin{array}{ccc}
   L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\
   -i \frac{d}{dx} \downarrow & & \downarrow \mathcal{M}_x \\
   L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R})
\end{array}$$

where $\mathcal{F}$ denotes the Fourier transform. Let $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ be the operator of “multiplication by $\frac{x}{\sqrt{1+x^2}}$”. Then

$$\left( \mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} f \right)(x) = \frac{x}{\sqrt{1+x^2}} f(x),$$

and $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ is a bounded operator

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}: L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Now, $T$ is the unique bounded operator $T: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that there is commutativity in the diagram

$$\begin{array}{ccc}
   L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\
   T \downarrow & & \downarrow \mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} \\
   L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R})
\end{array}$$

Then

$$(L^2(\mathbb{R}), \psi, \pi, T) \in \mathcal{E}^1_G(\mathbb{R}).$$

**Definition 1.32. Equivariant even $K$-homology** $K^0_G(A)$ of a $G$-C*-algebra $A$ is the group of homotopy classes of quadruples $(\mathcal{H}, \psi, T, \pi)$, where $(\mathcal{H}, \psi, \pi)$ is a covariant representation of $A$, and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\pi(g) T - T \pi(g) \in \mathcal{K}(\mathcal{H}), \quad \psi(a) T - T \psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a) (1 - T^* T) \in \mathcal{K}(\mathcal{H}), \quad \psi(a) (1 - TT^*) \in \mathcal{K}(\mathcal{H})$$

for all $g \in G, a \in A$.

$$K^0_G(A) = \{(\mathcal{H}, \psi, \pi, T)\} / \sim$$

If $A, B$ are $G$-C*-algebras, and $\varphi: A \to B$ is a $G$-equivariant *-homomorphism, then $\varphi^*: \mathcal{E}^j_G(B) \to \mathcal{E}^j_G(A)$ for $j = 0, 1$ is given by

$$\varphi^*(\mathcal{H}, \psi, \pi, T) \mapsto (\mathcal{H}, \psi \circ \varphi, \pi, T).$$

Addition in $K^j_G(A)$ is direct sum

$$(\mathcal{H}, \psi, \pi, T) + (\mathcal{H}', \psi', \pi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', \pi \oplus \pi', T \oplus T'),$$

and the inverse is

$$-(\mathcal{H}, \psi, \pi, T) = (\mathcal{H}, \psi, \pi, -T).$$
1.6 Hilbert modules

Let $A$ be a C*-algebra. Recall that an element $a \in A$ is positive (notation: $a \geq 0$) if and only if there exists $b \in A$ such that $b^*b = a$.

**Definition 1.33.** A pre-Hilbert $A$-module is a right $A$-module $H$ with a given $A$-valued inner product $\langle - , - \rangle$ such that

$$
\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle,
\langle u, va \rangle = \langle u, v \rangle a,
\langle u, v \rangle = \langle v, u \rangle^*,
\langle u, v \rangle \geq 0 \quad \forall u \in A,
\langle u, u \rangle = 0 \iff u = 0
$$

for $u, v_1, v_1, v \in H$, $a \in A$.

**Definition 1.34.** A Hilbert $A$-module is a pre-Hilbert $A$-module $H$ which is complete in the norm

$$
\|u\| = \|\langle u, u \rangle\|^{1/2}
$$

**Example 1.35.** A Hilbert $C$-module is a Hilbert space (viewed as a right $C$-module).

If $H$ is a Hilbert $A$-module, and $A$ has unit $1_A$, then $H$ is a $C$-vector space with

$$
u \lambda = u(\lambda 1_A), \quad \lambda \in \mathbb{C}.
$$

Moreover, even if $A$ does not have a unit, then by using approximate identity in $A$, it is a $C$-vector space.

**Example 1.36.** Let $A$ be $C^*$-algebra. We define a Hilbert $A$-module structure on $H = A^n$ by

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n),$$

$$(a_1, \ldots, a_n)a = (a_1a, \ldots, a_na),$$

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1^*b_1 + a_2^*b_2 + \cdots + a_n^*b_n.$$
Example 1.38. Let $G$ be a locally compact Hausdorff second countable topological group. Fix a left-invariant Haar measure $dg$ for $G$. Let $A$ be a $G$-$C^*$-algebra. Denote

$$L^2(G, A) := \{f : G \to A \mid \int_G g^{-1} f(g)^* f(g) dg \text{ is norm-convergent in } A\}.$$ 

Then $L^2(G, A)$ is a Hilbert $A$-module with operations

$$(f + h)g = f(g) + h(g),$$
$$(fa)(g) = f(g)[ga],$$
$$\langle f, h \rangle = \int_G g^{-1} f(g)^* h(g) dg.$$ 

Definition 1.39. An $A$-module map $T : \mathcal{H} \to \mathcal{H}$ is adjointable if there exists an $A$-module map $T^* : \mathcal{H} \to \mathcal{H}$ with

$$\langle Tu, v \rangle = \langle u, T^* v \rangle$$

for all $u, v \in \mathcal{H}$.

If $T^*$ exists, then it is unique, and sup$_\|u\| = 1 \|Tu\| < \infty$. Set

$$\mathcal{L}(\mathcal{H}) := \{T : A \to A \mid \|T\| \text{ is adjointable}\}.$$ 

Then $\mathcal{L}(\mathcal{H})$ is a $C^*$-algebra with operations

$$(T + S)u = Tu + Su$$
$$(ST)(u) = S(Tu)$$
$$(T\lambda)u = (Tu)\lambda$$
$$\|T\| = \sup_{\|u\| = 1} \|Tu\|$$

for $u \in \mathcal{H}, \lambda \in \mathbb{C}$.

1.7 Reduced crossed product

Let $A$ be a $G$-$C^*$-algebra. Denote

$$C_c(G, A) = \{f : G \to A \mid f \text{ is continuous and has compact support}\}$$

Then $C_c(G, A)$ is an algebra with operations

$$(f + h)(g) = f(g) + h(g)$$
$$(f\lambda)(g) = f(g)\lambda$$
$$(f * h)(g_0) = \int_G f(g)[gh(g^{-1}g_0)] dg$$

for $g, g_0 \in G, \lambda \in \mathbb{C}$. The operation $*$ is the twisted convolution. There is an injection of algebras $C_c(G, A) \to \mathcal{L}(L^2(G, A))$.

$$f \mapsto T_f, \quad T_f(u) = f * u$$

$$(f * u)(g_0) = \int_G f(g)[gu(g^{-1}g_0)] dg.$$
**Definition 1.40.** The reduced crossed product C*-algebra $C^r(G,A)$ is the completion of $C_c(G,A)$ in $L(L^2(G,A))$ with respect to the norm $\|f\| = \|T_f\|$.

**Example 1.41.** Let $G$ be a finite group and $A$ a $G$-C*-algebra. Assume that each $g \in G$ has mass 1. Then $C^*_r(G,A) = \{ \sum_{\gamma \in \Gamma} a_\gamma [\gamma] \mid a_\gamma \in A \}$ with the following operations

$$
\left( \sum_{\gamma \in \Gamma} a_\gamma [\gamma] \right) + \left( \sum_{\gamma \in \Gamma} b_\gamma [\gamma] \right) = \sum_{\gamma \in \Gamma} (a_\gamma + b_\gamma) [\gamma]
$$

$$(a_\gamma [\gamma])(b_\beta [\beta]) = a_\alpha (ab_\beta)_\alpha [\alpha \beta]
$$

$$
\left( \sum_{\gamma \in \Gamma} a_\gamma [\gamma] \right)^* = \sum_{\gamma \in \Gamma} (\gamma^{-1} a^*_\gamma) [\gamma^{-1}]
$$

$$
\left( \sum_{\gamma \in \Gamma} a_\gamma [\gamma] \right) \lambda = \sum_{\gamma \in G} (a_\lambda \gamma) [\gamma]
$$

for $\gamma \in G$, $\lambda \in \mathbb{C}$.

Let $X$ be a locally compact $G$-space. Then $C_0(X)$ is a $G$-C*-algebra with

$$(gf)(x) = f(g^{-1}x), \quad f, g \in C_0(X), \quad g \in G, \quad x \in X.$$  

We will denote $C^*_r(G,C_0(X))$ by $C^*_r(G,X)$. We ask about the K-theory of this C*-algebra.

If $G$ is compact, then $K_j(C^*_r(G,X))$ is the Atiyah-Segal group $K^j_G(X)$, $j = 0, 1$. Hence for $G$ non-compact $K_j(C^*_r(G,X))$ is the natural extension of the Atiyah-Segal theory to the case when $G$ is non-compact.

We say that the $G$-space is $G$-compact if and only if the quotient space $X/G$ is compact.

If $X$ is a proper $G$-compact $G$-space, then an equivariant C-vector bundle $E$ on $X$ determines an element $[E] \in K_0(C^*_r(G,X))$.

**Theorem 1.42** (W. Lück, B. Oliver). If $\Gamma$ is a (countable) discrete group and $X$ is a proper $\Gamma$-compact $\Gamma$-space, then $K_0(C^*_R(\Gamma,X))$ is the Grothendieck group of $\Gamma$-equivariant C-vector bundles on $X$.

### 1.8 Topological K-theory of $\Gamma$

Consider pairs $(M,E)$ such that $M$ is a $C^\infty$ manifold without boundary, with a given smooth proper co-compact action of $\Gamma$ and a given $\Gamma$-equivariant Spin$^c$-structure, and $E$ is a $\Gamma$-equivariant vector bundle on $M$. We introduce an equivalence relation on such pairs, which is generated by three elementary steps

- **Bordism**
- **Direct sum - disjoint union**
- **Vector bundle modification**
Then we define **topological K-theory** of $\Gamma$ as

$$K^\text{top}_0(\Gamma) \oplus K^\text{top}_1(\Gamma) = \{(M, E)\}/\sim.$$  

Addition will be disjoint sum

$$(M, E) + (M', E') = (M \cup M', E \cup E').$$

The main result of this section is:

**Theorem 1.43** (P. Baum, N. Higson, T. Schick). The map

$$\tau: K^\text{top}_j(\Gamma) \rightarrow K^\Gamma_j(\mathbb{E})$$

is an isomorphism for $j = 0, 1$.

We will describe the equivalence relation $\sim$ in details. We say that $(M, E)$ is **isomorphic** to $(M', E')$ if and only if there exist a $\Gamma$-equivariant diffeomorphism $\psi: M \rightarrow M'$ preserving the $\Gamma$-equivariant Spin$^c$-structures on $M$, $M'$ with $\psi^*E' \simeq E$. The equivalence relation is generated by three elementary steps:

- **Bordism:** we say that $(M_0, E_0)$ is bordant to $(M_1, E_1)$ if and only if there exists $(W, E)$ such that
  1. $W$ is a $C^\infty$ manifold with boundary, with a given smooth proper co-compact action of $\Gamma$
  2. $W$ has a given $\Gamma$-equivariant Spin$^c$-structure
  3. $E$ is a $\Gamma$-equivariant vector bundle on $W$
  4. $(\partial W, E|_{\partial W}) \simeq (M_0, E_0) \cup (-M_1, E_1)$.

- **Direct sum - disjoint union:** if $E, E'$ are $\Gamma$-equivariant vector bundles on $M$, then
  $$(M, E) \cup (M, E') \sim (M, E \oplus E').$$

- **Vector bundle modification:** let $F$ be a $\Gamma$-equivariant Spin$^c$ vector bundle on $M$. Assume that for every fiber $F_p$ we have $\dim_\mathbb{R}(F_p) = 0 \mod 2$. Take a one-dimensional $\Gamma$-equivariant trivial bundle $1 = M \times \mathbb{R}$, $\gamma(p, t) = (\gamma p, t)$. Let $S(F \oplus 1)$ be the unit sphere bundle of $F \oplus 1$. $F \oplus 1$ is a $\Gamma$-equivariant Spin$^c$ vector bundle with odd dimensional fibers. Let $\Sigma$ be the spinor bundle for $F \oplus 1$

  $$\pi: \text{Cl}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \rightarrow \Sigma_p.$$  

 Decompose $\pi^*\Sigma = \beta_+ \oplus \beta_-$. Then

  $$(M, E) \sim (S(F \oplus 1), \beta_+ \otimes \pi^*E).$$
1.9 **KK-theory**

Let $A$ be a C*-algebra, $\mathcal{H}$ a Hilbert module, $u, v \in \mathcal{L}(\mathcal{H})$. Denote

$$\theta_{u,v} \in \mathcal{L}(\mathcal{H}), \quad \theta_{u,v}(\xi) = u(v, \xi), \quad \theta_{u,v}^* = \theta_{v,u}.$$

The $\theta_{u,v}$ are the **rank one** operators on $\mathcal{H}$. A **finite rank** operator on $\mathcal{H}$ is any $T \in \mathcal{L}(\mathcal{H})$ such that $T$ is a finite sum of $\theta_{u,v}$.

$$T = \theta_{u_1,v_1} + \theta_{u_2,v_2} + \ldots + \theta_{u_n,v_n}.$$

The compact operators $\mathcal{K}(\mathcal{H})$ are defined as the norm closure in $\mathcal{L}(\mathcal{H})$ of the space of finite rank operators. It is an ideal in $\mathcal{L}(\mathcal{H})$.

We say that $\mathcal{H}$ is **countably generated** if in $\mathcal{H}$ there is a countable (or finite) set such that the $A$-module generated by this set is dense in $\mathcal{H}$.

Let $A, B$ be C*-algebras, $\varphi : A \to B$ a *-homomorphism, and $\mathcal{H}$ a Hilbert $A$-module. We will define $\mathcal{H} \otimes_A B$ which will be a Hilbert $B$-module. First form the algebraic tensor product $\mathcal{H} \otimes_A B$. It is a right $B$-module

$$(h \otimes b)b' = h \otimes bb', \quad h \in \mathcal{H}, \quad b, b' \in B.$$

Now define $B$-valued inner product $\langle -, - \rangle$ on $\mathcal{H} \otimes_A B$ by

$$\langle h \otimes b, h' \otimes b' \rangle = b^* \varphi((h, h')) b'.$$

Set

$$N := \{\xi \in \mathcal{H} \otimes_A B \mid \langle \xi, \xi \rangle = 0\}.$$

It is a $B$-submodule of $\mathcal{H} \otimes_A B$, and $\mathcal{H} \otimes_A B/N$ is a pre-Hilbert $B$-module.

**Definition 1.44.** $\mathcal{H} \otimes_A B$ is the completion of $\mathcal{H} \otimes_A B/N$.

Let $A, B$ be separable C*-algebras, $\mathcal{E}^1(A, B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a countably generated Hilbert $B$-module, $\psi : A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism, $T \in \mathcal{L}(\mathcal{H})$ is such that

$$T = T^*$$

$$\psi(a)(I - T^2) \in \mathcal{K}(\mathcal{H})$$

$$\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$.

We say that $(\mathcal{H}_0, \psi_0, T_0), (\mathcal{H}_1, \psi_1, T_1) \in \mathcal{E}^1(A, B)$ are **isomorphic** if there exists an isomorphism of Hilbert $B$-modules $\Phi : \mathcal{H}_0 \to \mathcal{H}_1$ with

$$\Phi \psi_0(a) = \psi_1(a) \Phi, \quad \text{for all } a \in A, \Phi T_0 = T_1 \Phi.$$

Let $A, B, D$ be separable C*-algebras, $\varphi : B \to D$ a *-homomorphism. There is an induced map

$$\varphi_* : \mathcal{E}^1(A, B) \to \mathcal{E}^1(A, D),$$

$$\varphi_*(\mathcal{H}, \psi, T) = (\mathcal{H} \otimes_B D, \psi \otimes_B I, T \otimes_B I),$$

where $I$ is the identity operator of $D$. 16
Consider two maps $\rho_0, \rho_1: C([0,1], B) \to B$, $\rho_0(f) = f(0)$, $\rho_1(f) = f(1)$. We say that $(\mathcal{H}_0, \psi_0, T_0), (\mathcal{H}_1, \psi_1, T_1) \in \mathcal{E}^1(A, B)$ are homotopic if there exists $(\mathcal{H}, \psi, T) \in \mathcal{E}^1(A, C([0,1], B))$ with $(\rho_j)_*(\mathcal{H}, \psi, T) \simeq (\mathcal{H}_j, \psi_j, T_j)$.

For the even case, consider $\mathcal{E}^0(A, B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a countably generated Hilbert $B$-module, $\psi: A \to \mathcal{L}(\mathcal{H})$ is a $\ast$-homomorphism, and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H})$$
$$\psi(a)(I - T^*T) \in \mathcal{K}(\mathcal{H})$$
$$\psi(a)(I - TT^*) \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$.

**Definition 1.45.** We define the **KK-theory** of $A, B$ as

$$\text{KK}^0(A, B) := \mathcal{E}^0(A, B)/\sim$$
$$\text{KK}^1(A, B) := \mathcal{E}^1(A, B)/\sim$$

where the relation $\sim$ is homotopy. $\text{KK}^j(A, B)$ is an abelian group

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$
$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, T^*)$$.

### 1.10 Equivariant KK-theory

Let $A$ be a $G$-C*-algebra.

**Definition 1.46.** A $G$-Hilbert $A$-module is a Hilbert $A$-module $\mathcal{H}$ with a given continuous action $G \times \mathcal{H} \to \mathcal{H}$, $(g, v) \mapsto gv$ such that

$$g(u + v) = gu + gv$$
$$g(ua) = (gu)(ga)$$
$$\langle gu, gv \rangle = g\langle u, v \rangle$$

for $u, v \in \mathcal{H}$, $g \in G$, $a \in A$. Continuity here means that for each $u \in \mathcal{H}$, $g \mapsto gu$ is a continuous map $G \to \mathcal{H}$.

For each $g \in G$, denote by $L_g$ the map $L_g: \mathcal{H} \to \mathcal{H}$, $L_g(v) = gv$. Note that $L_g$ might not be in $\mathcal{L}(\mathcal{H})$. But if $T \in \mathcal{L}(\mathcal{H})$, then $L_gTL_g^{-1} \in \mathcal{L}(\mathcal{H})$. Thus $\mathcal{L}(\mathcal{H})$ is a $G$-C*-algebra with $gT = L_gTL_g^{-1}$.

**Example 1.47.** If $A$ is a $G$-C*-algebra, $n$ positive integer. Then $A^n$ is a $G$-Hilbert $A$-module with $g(a_1, a_2, \ldots, a_n) = (ga_1, ga_2, \ldots, a_n)$.

Let $A, B$ be separable $G$-C*-algebras, $\mathcal{E}^1(A, B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a $G$-Hilbert $B$-module (countably generated), $\psi: A \to \mathcal{L}(B)$ is a $\ast$-homomorphism with

$$\psi(ga) = g\psi(a), \ g \in G, a \in A,$$
and \( T \in \mathcal{L}(\mathcal{H}) \) is such that

\[
T = T^* \\
gT - T \in \mathcal{K}(\mathcal{H}) \\
\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}) \\
\psi(a)(I - T^2) \in \mathcal{K}(\mathcal{H})
\]

for all \( g \in G, \ a \in A \).

In the even case we take \( E_0(A,B) = \{(\mathcal{H},\psi,T)\} \), where \( \mathcal{H} \) is a \( G \)-Hilbert \( B \)-module (countably generated), \( \psi: A \to \mathcal{L}(B) \) is a \(*\)-homomorphism with

\[
\psi(ga) = g\psi(a), \ g \in G, \ a \in A,
\]

and \( T \in \mathcal{L}(\mathcal{H}) \) is such that

\[
gT - T \in \mathcal{K}(\mathcal{H}) \\
\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}) \\
\psi(a)(I - T^*T) \in \mathcal{K}(\mathcal{H}) \\
\psi(a)(I - TT^*) \in \mathcal{K}(\mathcal{H})
\]

for all \( g \in G, \ a \in A \).

**Definition 1.48.** We define the **equivariant KK-theory** of \( A,B \) as

\[
KK_0^G(A,B) := E_0(A,B)/\sim \\
KK_1^G(A,B) := E_1(A,B)/\sim
\]

where the relation \( \sim \) is homotopy. \( KK_j^G(A,B) \) is an abelian group

\[
(\mathcal{H},\psi,T) + (\mathcal{H}',\psi',T') = (\mathcal{H} \oplus \mathcal{H}',\psi \oplus \psi',T \oplus T') \\
-(\mathcal{H},\psi,T) = (\mathcal{H},\psi,T^*)
\]

**1.11 K-theory of the reduced group \( C^* \)-algebra**

If a compact group \( G \) acts on \( \mathbb{C} \) by a \( C^* \)-automorphisms, then it must act trivially, since \( \mathbb{C} \) has no nontrivial \(*\)-automorphisms. We will prove the following:

**Theorem 1.49.** For a compact group \( G \) there is an isomorphism

\[
K_0(C^*_r(G)) \simeq R(G).
\]

The key element in the proof is the Peter Weyl theorem:

**Theorem 1.50 (Peter Weyl).** If \( G \) is a compact, Hausdorff, second countable unitary representation of \( G \), then every irreducible unitary representation of \( G \) is finite dimensional.

**Proof.** Let \( \rho: G \to U(\mathcal{H}) \) be an irreducible representation on a separable Hilbert space \( \mathcal{H} \).
Choose a projection \( p \) on \( \mathcal{H}, \ p \neq 0, \ p = p^* \) with finitely dimensional range. Let

\[
T := \int_G \rho(g)pp(g)^*dg,
\]

where \( dg \) is a Haar measure. Then
\begin{itemize}
\item $T$ commutes with $\rho(g)$ for all $g \in G$,
\item $T = T^*$, $T \geq 0$, $T \neq 0$,
\item $T$ is compact operator, $T \in \mathcal{K}(\mathcal{H})$.
\end{itemize}

The structure theorem for compact selfadjoint positive operators gives

$$\text{sp}(T) := \{a_n \in \mathbb{R} \mid a_n \to 0\},$$

where each $a_n$ is an eigenvalue with finitely dimensional eigenspace. In particular any compact selfadjoint operator has finite dimensional eigenspace. For $T$ this eigenspace has to be preserved by the group action, so $\rho$ has to be finitely dimensional if it is irreducible.

Proof. (of Theorem 1.49) Notice that for compact group $C^*_r(G) = C^*(G)$ (there is only one C*-algebra for a compact group). Irreducible unitary representations of $G$ (up to equivalence) form a countable set. There is a C*-isomorphism

$$C^*(G) \simeq \bigoplus_{\sigma \in \text{Irrep}(G)} A_{\sigma},$$

where each $A_\sigma$ is a finitely dimensional C*-algebra, which is isomorphic to $M_n(\mathbb{C})$, $n = \dim \sigma$. Hence

$$K_j(C^*(G)) \simeq \bigoplus_{\sigma \in \text{Irrep}(G)} K_j(A_\sigma) \simeq \begin{cases} R(G) & \text{for } j = 0, \\ 0 & \text{for } j = 1. \end{cases}$$

For a compact group $G$ we have the map

$$\mu : K^G_j(EG) \to K_j(C^*_r(G)).$$

The elements of $K^G_j(EG)$ can be viewed as generalized elliptic operators on $EG$. The map $\mu$ assigns to such a generalized elliptic operator its index

$$\mu(\mathcal{H}, \psi, T, \pi) = \ker T - \text{coker } T.$$

1.12 $KK^0_G(\mathbb{C}, \mathbb{C})$

If $G$ is a compact group then $EG = \text{pt}$ and $K_0(C^*_r(G)) = R(G)$ - the representation ring of $G$. We obtain $R(G)$ as a Grothendieck group of the category of finite dimensional (complex) representations of $G$. It is a free abelian group with one generator for each distinct (i.e. nonequivalent) irreducible representation of $G$.

**Theorem 1.51.** For a compact group $G$ there is an isomorphism

$$K_G(\mathbb{C}, \mathbb{C}) \simeq R(G).$$

Proof. Given $(\mathcal{H}, \psi, T, \pi) \in E^0_G(\mathbb{C})$ within the equivalence relation on $E^0_G(\mathbb{C})$ we may assume that

$$T \pi(g) - \pi(g)T = 0,$$

(1.1)
because we can average $T$ over the compact group $G$

$$T' := \int_G \pi(g)T\pi(g)^*dg = 0,$$

$$T - T' = T - \int_G \pi(g)T\pi(g)^*dg$$

$$= \int_G (T - \pi(g)T\pi(g)^*)dg \in \mathcal{K}(H),$$

because $\int_G Tdg = T$ since we normalize Haar measure.

Furthermore we can assume that

$$\psi(\lambda) = \lambda \text{Id}. \quad (1.2)$$

Indeed, $\psi : \mathbb{C} \rightarrow B(H)$ is a \ast\-homomorphism, and $\psi(1)$ is a selfadjoint projection. For all $\lambda \in \mathbb{C}$

$$\psi(\lambda) = \lambda \psi(1), \quad p := \psi(1).$$

$H$ splits into $pH \oplus (1-p)H$, and

$$Tp - pT \in \mathcal{K}(H), \quad T(1-p) - (1-p)T \in \mathcal{K}(H).$$

Compare $T$ to $pTp \oplus (1-p)T(1-p)$, to see that on $(1-p)H$ $\psi$ is 0.

The only nontrivial condition on $(H, \psi, T, \pi)$ is

$$I - T^*T \in \mathcal{K}(H),$$

$$I - TT^* \in \mathcal{K}(H).$$

These conditions imply that $T$ is Fredholm, that is

$$\dim_{\mathbb{C}}(\ker T) < \infty,$$

$$\dim_{\mathbb{C}}(\coker T) < \infty.$$

The spaces $\ker T$ and $\coker T$ are finite dimensional representations of $G$. We have

$$\mu(H, \psi, T, \pi) = \ker T - \coker T \in R(G).$$

First we will prove the surjection of $K_G(\mathbb{C}, \mathbb{C}) \rightarrow R(G)$. Let $V \in R(G)$ be finitely dimensional irreducible unitary representation. Consider countable direct sum $\bigoplus V$ and $\bigoplus \pi$. Let $T$ be a shift

$$(v_1, v_2, \ldots) \mapsto (v_2, v_3, \ldots).$$

Then $\ker T = V$ (first copy), and $\coker T = 0$.

The homomorphism $K_G(\mathbb{C}, \mathbb{C}) \rightarrow R(G)$ is well defined and injective. Indeed, consider irreducible representation $V \in R(G)$. There is a canonical decomposition into isotypical components

$$V = n_1V_1 \oplus n_2V_2 \oplus \ldots n_kV_k.$$