Lectures on Homology of Symbols

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1 The algebra of classical symbols
1.1 Local definition of the algebra of symbols ......................... 3
1.2 Classical pseudodifferential operators ............................. 5
1.3 Statement of results ............................................. 7
1.4 Derivations of the de Rham algebra ............................... 7
1.5 Koszul-Chevalley complex ...................................... 12
1.6 A relation between Hochschild and Lie algebra homology .......... 13
1.7 Poisson trace .................................................... 15
   1.7.1 Graded Poisson trace ..................................... 17
1.8 Hochschild homology ........................................... 18
1.9 Cyclic homology ............................................... 21
   1.9.1 Further analysis of spectral sequence ....................... 25
   1.9.2 Higher differentials ..................................... 30

A Topological tensor products ........................................ 32

B Spectral sequences ................................................. 34
   B.1 Spectral sequence of a filtered complex ...................... 34
   B.2 Examples .................................................. 37
Chapter 1

The algebra of classical symbols

1.1 Local definition of the algebra of symbols

Let $X$ be a $C^\infty$-manifold (not necessarily compact), and $E$ a vector bundle on $X$. Consider a coordinate patch

$$ f_U : U \rightarrow X, \quad U \subset \mathbb{R}^n. $$

The cotangent bundle $T^*X \rightarrow X$ pulls back to $U$

$$ T_0^*U \longrightarrow T^*U \longrightarrow T^*X $$

$$ \pi \downarrow \quad \downarrow \quad \downarrow f_U \Rightarrow X $$

The bundle $T_0^*U$ is defined as $T^*U \setminus U$. There is an isomorphism

$$ T_0^*U \xrightarrow{\sim} U \times \mathbb{R}\times U \subset \mathbb{R}^n \times \mathbb{R}^n $$

$$ \pi \downarrow \quad \downarrow U $$

Using it we can denote the coordinates on $T_0^*U$ by $(u, \xi)$, where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

To each open set $U$ we associate a section $a^U := \sum_{j=0}^{\infty} a^U_j$, where each $a^U_j$ is a section of the bundle $\text{End}(\pi^*f_U^*E)$, where

$$ \pi^*f_U^*E \longrightarrow f_U^*E \longrightarrow E $$

$$ T_0^*U \quad \pi \quad U \quad f_U \quad X $$

More precisely by $a^U_{m-j}$ we denote the homogeneous part of degree $m - j$

$$ a^U_{m-j} \in C^\infty(T_0^*U, \text{End}(\pi^*f_U^*E))(m - j). $$

There is a natural action of $\mathbb{R}_+$ on $T_0^*X$ given by $t \cdot (u, \xi) := (u, t\xi)$. The infinitesimal action is provided by the Euler field

$$ \Xi = \sum_{i=1}^{n} \xi_i \partial_{\xi_i}. $$
The homogeneity condition for \( a^U_{m-j} \) is given by \( a^U_{m-j}(u, t\xi) = t^{m-j}a^U_{m-j}(u, \xi) \).

The section \( a^U \) belongs to the product

\[
\prod_{j=0}^{\infty} C^\infty(T_0^*U, \text{End}(\pi^*f^*_UE))(m-j)
\]

which has a natural structure of Fréchet space. With the norm

\[
|\xi| := \sqrt{\xi_1^2 + \cdots + \xi_n^2}
\]

we can write

\[
|\xi|^{-m}a^U_{m-j} \in C^\infty(T_0^*U, \text{End}(\pi^*f^*_UE))(0) \simeq C^\infty(S^*U, \text{End}(\pi^*f^*_UE)),
\]

where \( S^*U \) is the cosphere bundle \( T_0^*U/\mathbb{R}^n_+ \mathbb{R}_+U \). The cotangent bundle \( T^*X \to X \) is canonically oriented and \( S^*X \) is canonically oriented (even though we do not have the orientation on \( X \)). Now \( S^*U \) is a canonically oriented \((2n-1)\)-manifold and \( S^*U \simeq U \times S^{n-1} \).

The sections \( a^U \) are given locally, so we need a compatibility condition. We need a composition law such that it will depend on all jets, not only on 1-jets as usual composition.

\[
a^U \circ_b b^U : \sum_\alpha \delta_\alpha^a a^U D^{|\alpha|}_a b^U
\]

where \( \alpha = (a_1, \ldots, a_n), \quad a_i \in \mathbb{N} \)

\[
D_{u_i} := \frac{1}{i!} \partial_{u_i}, \quad D^{|\alpha|}_a = \frac{1}{\alpha!} D^\alpha_a = \frac{1}{\alpha! |\alpha|!} \partial^\alpha_u.
\]

If \( a^U \) is of order \( m \), \( b^U \) of order \( m' \) using the notation for classical symbols

\[
\mathcal{CS}^m_U(U, E) := \prod_{j=0}^{\infty} (T^*_jU, \text{End}(\pi^*f^*_UE))(m-j)
\]

we can write

\[
\circ_u : \mathcal{CS}^m_U(U, E) \times \mathcal{CS}^{m'}_{U'}(U, E) \to \mathcal{CS}^{m+m'}_{U'}(U, E), \quad m, m' \in \mathbb{C}.
\]

Now suppose we have two open sets \( U, V \in \mathbb{R}^n \) such that the images of charts \( f_U : U \to X, \)

\( f_V : V \to X \) have nonempty intersection \( f(U) \cap f(V) \). Denote

\[
U' := f^{-1}_U(f(U) \cap f(V)), \quad V' := f^{-1}_V(f(U) \cap f(V)),
\]

\[
f_{UV} := f^{-1}_U \circ f_V : V' \to U'.
\]

For a smooth map \( f : X \to Y \) there are induced maps

\[
Tf : TC \to TY, \quad (Tf)_x : (TX)_x \to (TY)_{f(x)},
\]

\[
T^*f : T^*X \to T^*Y, \quad (T^*f)_x : (T^*X)_x \leftarrow (T^*Y)_{f(x)}.
\]

Assume that \( Tf \) is invertible

\[
((Tf)_x)^{-1} : (TX)_x \to (TY)_{f(x)}
\]
Define a map
\[X \times TX \to Y \times TY, \quad (x, v) \mapsto (f(x), (Tf)_x(v)),\]
\[X \times T^*X \to Y \times T^*Y, \quad (x, \xi) \mapsto (f(x), ((Tf)^*)_x^{-1}(\xi)).\]

Now comes the question, to what extend \(a^V\) and \(((Tf)^*)^*a^U\) agree? We have
\[
(a^V) = (T^*f)^*((a^U) + (\text{arbitrary high order correction terms}))
\]
\[= (T^*f)_{UV}^*(\sum a_\lambda \partial x^a),\]
where
\[
\psi_a(u, \xi) = D_x^a e^{i(j_u^{-1}(z), (Tf^U)_x(\xi))}|_{z = u, v = (f^V_{UV} \circ f_U)(u)},
\]
so \(j_u^{-1}\) vanishes up to second order at point \(u \in U\). The \(\psi_a(u, \xi)\) are scalar valued functions on coordinate charts. They do not depend on symbols, only on manifold.

In the whole notes we will be using a projective tensor product of topological vector spaces described in the appendix (A).

The product
\[\text{CS}^m(X, E) \times \text{CS}^{m'}(X, E) \to \text{CS}^{m+m'}(X, E)\]
of Frechet spaces is associative. Define the algebra of symbols as
\[\text{CS}(X, E) : = \bigcup_{m \in \mathbb{Z}} \text{CS}^m(X, E).\]

Let \(a := \{a^U\}_{f_U : U \to X}\). The topology on \(\text{CS}(X, E)\) is defined as follows. We say that the net \(\{a_\lambda\}\) converges to a symbol \(a\) if for any \(m \in \mathbb{C}\) there exists \(\lambda_0\) such that \(a_\lambda - a \in \text{CS}^m(X, E)\) for all \(\lambda \geq \lambda_0\).

The subalgebra \(\text{CS}^0(X, E)\) is a Frechet algebra, and \(\text{CS}^{-j}(X, E), j \in \mathbb{Z}_+\) is a two sided ideal in \(\text{CS}^0(X, E)\).

**Remark 1.1.** The multiplication
\[\text{CS}^m(X, E) \otimes \text{CS}(X, E) \to \text{CS}(X, E)\]
is not continuous in both arguments.

### 1.2 Classical pseudodifferentials operators

Let \(A : \text{C}^\infty_c(X, E) \to \text{C}^\infty(X, E)\) be a pseudo differential operator. For a chart \(f_U : U \to X\) there is an operator
\[f_U^*A : \text{C}^\infty_c(U, f_U^*E) \to \text{C}^\infty(U, f_U^*E)\]
We can define it for \(\varphi \in \text{C}^\infty_c(U, f_U^*E)\) as follows. First take \((\varphi \circ f_U^{-1})|_{f_U(\text{supp} \varphi)}\) and then extend by 0, apply \(A\) and pullback, as in the following diagram
\[
\begin{array}{c}
\text{C}^\infty_c(X, E) \xrightarrow{A} \text{C}^\infty(X, E) \\
(f_U)| \downarrow \quad \quad \downarrow f_U \\
\text{C}^\infty_c(U, f_U^*E) \xrightarrow{f_U^*A} \text{C}^\infty(U, f_U^*E)
\end{array}
\]
Explicitly
\[
(f^u U_A) \varphi(u) = \int_{\mathbb{R}^n} \int_U e^{i(u-u',\xi)} \beta(u,u',\xi) \varphi(u') du' d\xi + (T \varphi)(u),
\]
where $\beta \in C^\infty(U \times T^* U, \text{End}(\pi^* f_U^* E))$ is called an amplitude,
\[
\beta(u,u',\xi) \sim \sum_{j=0}^{\infty} \beta_{m-j}(u,u',\xi),
\]
\[
\beta_{m-j}(u,u',t\xi) = t^{m-j} \beta(u,u',\xi),
\]
$T$ is a smoothing operator
\[
(T \varphi)(u) = \int_U K(u,u') \varphi(u') |du'|,
\]
and
\[
|du| = |du_1 \wedge \cdots \wedge du_n|, \quad d\xi = \frac{1}{(2\pi)^n} |d\xi_1 \wedge \cdots \wedge d\xi_n|.
\]

By $\text{CL}^m(X,E)$ we denote the space of classical pseudo differential operators, and by $\text{CL}^m_{\text{prop}}(X,E)$ the subset of operators which take functions with compact support into functions with compact support. For $A \in \text{CL}^m(X,E)$ there is a decomposition $A = A_{\text{prop}} + S$ into a proper part $A_{\text{prop}}$ and non proper smoothing part $S$. Define a Frechet space of arbitrary low order operators by
\[
\mathcal{L}^{-\infty}(X,E) := \bigcap_{m \in \mathbb{Z}} \text{CL}^m(X,E).
\]

There is an isomorphism
\[
\text{CL}^m(X,E)/\mathcal{L}^{-\infty}(X,E) \cong \text{CS}^m(X,E).
\]

Classical symbols have a product
\[
\text{CL}^m_{\text{prop}}(X,E) \times \text{CL}^{m'}_{\text{prop}}(X,E) \to \text{CL}^{m+m'-1}_{\text{prop}}(X,E), \quad m,m' \in \mathbb{C}.
\]

We define the algebra of classical symbols as
\[
\text{CL}(X,E) := \bigcup_{m \in \mathbb{Z}} \text{CL}^m(X,E).
\]

The space of smoothing operators $\mathcal{L}^\infty(X,E)$ is defined as a kernel
\[
\mathcal{L}^\infty(X,E) \rightarrow \text{CL}(X,E) \rightarrow \text{CS}(X,E)
\]
and if $X$ is closed it is isomorphic (non canonically) to the space of rapidly decaying matrices
\[
\mathcal{L}^{-\infty} = \{(a_{ij})_{i,j=1,...,\infty} \mid |a_{ij}|(i+j)^N \to 0, \text{ as } i+j \to \infty\}.
\]

This is the noncommutative orientation class of a closed manifold and index theorem is the way to state that. Index measures to what extend this sequence is not split.

The map
\[
\text{CL}(X,E)/\mathcal{L}^{-\infty}(X,E) \rightarrow \text{CS}(X,E)
\]
is defined as follows. For a classical pseudo differential operator

\[ A : C^\infty_c(X, E) \to C^\infty(X, E) \]

we take the amplitude

\[ \beta^{U}(u, u', \xi) \sim \sum_{j=0}^{\infty} \beta^{U}_{m-j}(u, u', \xi) \]

and then define \( a^{U} \in \text{CS}(X, E) \) by

\[ a^{U} := \left( e^{\sum_{i=1}^{n} \partial_{i} D_{u} \beta^{U}} \right) \bigg|_{u = u'}. \]

### 1.3 Statement of results

The main goal is to compute the Hochschild and cyclic homology of the algebra of symbols \( \text{CS}(X) \). Let \( T^{*}_{0} X = T^{*} X \setminus X \) and \( Y^c \) be the \( C^* \)-bundle over the cosphere bundle \( S^{*} X \) defined as

\[ Y^c := T^{*}_{0} X \times_{\mathbb{R}^+} \mathbb{C}^* \]

\[ S^{*} X \]

**Theorem 1.2.** There is a canonical isomorphism

\[ \text{HH}_{q}(\text{CS}(X)) \simeq H^{2n-q}_{\text{dR}}(Y^c). \]

Regarding cyclic homology, consider on \( \text{HC}_{q}^{\text{cont}}(\text{CS}(X)) \) the filtration by the kernels of the iterated \( S \)-map:

\[ \{0\} = S_{q0} \subset S_{q1} \subset \ldots \subset S_{qt} = \text{HC}_{q}(\text{CS}(X)), \]

where \( t = \left[ \frac{q}{2} \right] \) and \( S_{qr} := \ker S^{1+r} \cap \text{HC}_{q}(\text{CS}(X)) \).

**Theorem 1.3.** The canonical map

\[ I : \text{HH}_{*}(\text{CS}(X)) \to \text{HC}_{*}(\text{CS}(X)) \]

is injective. In particular

\[ \text{HC}_{qr}(\text{CS}(X)) = \text{gr}_{r}^{S} \text{HC}_{q}(\text{CS}(X)) := S_{qr}/S_{q,r-1} \]

is canonically isomorphic with

\[ H^{2n-q+2r}_{\text{dR}}(Y^c), \quad r = 0, 1, \ldots. \]

### 1.4 Derivations of the de Rham algebra

Let \( \mathcal{O} \) be a commutative \( k \)-algebra with unit, and \( k \) any commutative ring of coefficients. We define

\[ \Omega^{\bullet}_{\mathcal{O}/k} := \Lambda^{\bullet}_{\mathcal{O}} \Omega^{1}_{\mathcal{O}/k}, \]

where \( \Omega^{1}_{\mathcal{O}/k} \) can be defined in a three ways:
• Serre’s picture

$$\Omega^1_{O/k} := I_\Delta / I^2_\Delta,$$

where $$I_\Delta := \text{ker}(O^2 \to O).$$

• Hochschild picture

$$\Omega^1_{O/k} := O^{\otimes 2} / bO^{\otimes 3}.$$  

• Leibniz picture

$$\Omega^1_{O/k} := \frac{\mathcal{O}(df \mid f \in \mathcal{O})}{\mathcal{O}(d(f + g) - df - dg, \ dc = 0 \ (c \in k), \ d(fg) - fdg - gdf)}.$$  

The differential $$d : \mathcal{O} \to \Omega^1_{O/k}$$ is defined in those three pictures as follows

- $$f \mapsto df \mod I^2_\Delta = (1 \otimes f - f \otimes 1) \ mod \ I^2_\Delta$$ (Serre’s picture),
- $$f \mapsto df \mod bO^{\otimes 3} = (1 \otimes f - f \otimes 1) \ mod \ bO^{\otimes 3}$$ (Hochschild picture),
- $$f \mapsto df$$ (Leibniz picture).

The derivation $$d_\Delta : \mathcal{O} \to I_\Delta \subset \mathcal{O} \otimes \mathcal{O}$$ is universal in the sense that if we have an $$\mathcal{O}$$-bimodule $$M$$ and a derivation $$\delta : \mathcal{O} \to M$$, then there exists a unique $$\mathcal{O}$$-bimodule map $$\bar{\delta}$$ such that the following diagram commutes

$$\begin{array}{ccc}
M & \xrightarrow{\delta} & M \\
\downarrow \delta & & \downarrow \delta \\
\mathcal{O} & \xrightarrow{d} & \mathcal{O}/I^2_\Delta \\
\downarrow \delta_\Delta & & \downarrow \delta_\Delta \\
I_\Delta & & I_\Delta
\end{array}$$

Let $$\text{Der}^m(\Omega^\bullet) = \text{Der}^m_{\text{fr}}(\Omega^\bullet)$$ denote the algebra of degree $$m$$ derivations, and

$$\text{Der}^\bullet(\Omega^\bullet) := \bigoplus_{m \in \mathbb{Z}} \text{Der}^m(\Omega^\bullet).$$

If $$\eta$$ is of degree $$p$$ and $$\zeta$$ of degree $$q$$, then for $$\delta \in \text{Der}^m(\Omega^\bullet)$$ we have

$$\delta(\eta \wedge \zeta) = \delta(\eta) \wedge \zeta + (-1)^p \eta \wedge \delta(\zeta).$$

$$\delta : \Omega^p \to \Omega^{p+m}.$$  

Furthermore $$\text{Der}^\bullet(\Omega^\bullet)$$ is a super Lie algebra, that is the commutators satisfy the super Jacobi identity

$$0 = [[a, b], c] + (-1)^{|a|(|b|+|c|)} [[b, c], a] + (-1)^{|c|(|a|+|b|)} [[c, a], b].$$

Denote $$\delta_p := \delta|_{\Omega_p}.$$

**Proposition 1.4.** The set $$\text{Der}^m(\Omega^\bullet)$$ is naturally identified with the set of pairs $$(\delta_0, \delta_1)$$, where

$$\delta_0 : \mathcal{O} \to \Omega^m$$

is a $$k$$-linear derivation of $$\mathcal{O}$$ with values in $$\Omega^m$$,

$$\delta_1 : \Omega^1 \to \Omega^{m+1}.$$
is a $k$-linear map such that
\[
\delta_1(f\alpha) = \delta_0(f) \wedge \alpha + f\delta_1(\alpha).
\]
and
\[
\delta_1(\alpha)f - (-1)^{m+1}f\delta_1(\alpha) = 0,
\]
that is the super commutator $[\delta_1(\alpha), \alpha] = 0$.

Any derivation of degree $m$ is uniquely determined by $\delta_0$ and $\delta_1$. Thus $\text{Der}^m(\Omega^*) = 0$ for $m < -1$.

For $\delta_0 = 0$ we have
\[
\delta(f\alpha_1 \wedge \cdots \wedge \alpha_p) = \sum_{i=1}^p (-1)^{m(i-1)}f\alpha_1 \wedge \cdots \wedge \delta_1(\alpha_i) \wedge \cdots \wedge \alpha_p.
\]

Similarly for any $\phi \in \text{Hom}_O(\Omega^1, \Omega^{m+1})$ there exists a corresponding derivation
\[
\delta_\phi(f\alpha_1 \wedge \cdots \wedge \alpha_p) := \sum_{i=1}^p (-1)^{m(i-1)}f\alpha_1 \wedge \cdots \wedge \phi(\alpha_i) \wedge \cdots \wedge \alpha_p.
\]

**Example 1.5.** (The de Rham derivation) Let $d_0 = d : O \to \Omega^1$. Now we will give a construction of $d_1 : \Omega^1 \to \Omega^2$. Consider a $k$-linear pairing
\[
O \times O \to \Omega^2, \quad (f, g) \mapsto df \wedge dg
\]
\[
\begin{array}{c}
O \times O \to \Omega^2 \\
\downarrow \\
O \otimes_k O \\
\downarrow \\
(O \otimes_k O)/I_\Lambda^2
\end{array}
\]

Now we can take a restriction to $I_\Lambda/I_\Lambda^2 \subset (O \otimes_k O)/I_\Lambda^2$. Recall that $I_\Lambda$ consists of sums of terms of the form
\[
f_0d_\Lambda f_1 = f_0(1 \otimes f_1 - f_1 \otimes 1) = f_0 \otimes f_1 - f_0f_1 \otimes 1.
\]

Similarly $I_\Lambda^2$ consists of sums of terms of the form
\[
f_0d_\Lambda f_1d_\Lambda f_2 = f_0(1 \otimes f_1 - f_1 \otimes 1)(1 \otimes f_2 - f_2 \otimes 1) = f_0(1 \otimes f_1 f_2 + f_1 f_2 \otimes 1 - f_1 \otimes f_2 - f_2 \otimes f_1) = f_0 \otimes f_1 f_2 + f_0 f_1 f_2 \otimes 1 - f_0 f_1 \otimes f_2 - f_0 f_2 \otimes f_1
\]

The last expression maps to
\[
df_0 \wedge d(f_1 f_2) + d(df_0 f_1 f_2) \wedge d1 - d(f_0 f_1) \wedge df_2 - d(f_0 f_2) \wedge df_1 = df_0 \wedge ((df_1) f_2 + f_1 df_2) - (df_0 f_1) + f_0 df_1) \wedge df_2 - (df_0 f_2) + f_0 df_2) \wedge df_1 = f_2 df_0 \wedge df_1 + f_1 df_0 \wedge df_2 - f_1 df_1 \wedge df_2 - f_0 df_1 \wedge df_1 - f_2 df_0 \wedge df_1 = -f_0 df_1 \wedge df_2 - f_0 df_2 \wedge df_1
\]
\[
= 0.
\]
Proposition 1.6. Any derivation $\delta \in \text{Der}_k^m(\Omega^*)$ can be uniquely expressed as

$$[\delta_\varphi, d] + \delta_\psi$$

for $\varphi \in \text{Hom}_O(\Omega^1, \Omega^m), \psi \in \text{Hom}_O(\Omega^1, \Omega^{m+1})$.

Example 1.7. ($O$-linear derivation) For $m = -1 \text{ Der}^{-1}_k(\Omega^*) = \text{ Der}^{-1}_O(\Omega^*)$ and by restriction to $\Omega^1$

$$\text{ Der}^{-1}_O(\Omega^*) = \text{ Hom}_O(\Omega^1, \Omega^*).$$

If $O = O(X)$, then $\text{ Der}_k O = T X$.

Suppose that $\delta, \delta' \in \text{ Der}_k^m(\Omega^*)$ are such that

$$\delta_0 = \delta|_O = \delta'|_O = \delta'_0.$$

Then

$$\delta - \delta' \in \text{ Der}_O^m(\Omega^*) \quad O - \text{linear}.$$

Suppose that we have a derivation $D \in \text{ Der}_k^1(\Omega^*)$. Then for any $\varphi \in \text{ Hom}_O(\Omega^1, \Omega^m)$ there is a $\delta_\varphi \in \text{ Der}_O^{m-1}(\Omega^*)$ and

$$[\delta_\varphi, D] \in \text{ Der}_O^m(\Omega^*)$$

$$[\delta_\varphi, D]|_0 = \delta_\varphi D = \varphi \circ D = d(\text{the de Rham derivation})$$

If there exists $d_1 : \Omega^1 \to \Omega^2$, $k$-linear and satisfying

$$d_1(f \alpha) = df \wedge \alpha + f d\alpha,$$

then there exists a derivation $d \in \text{ Der}_k^1(\Omega^*)$.

There is a natural identification between $O$-modules

$$\begin{array}{ccc}
\text{Der}(O, \Omega^m) & \longrightarrow & \text{ Hom}_O(\Omega^1, \Omega^m) \\
\downarrow & & \downarrow \\
\text{ Der}^{m-1}_O(\Omega^*) & & 
\end{array}$$

Let $\eta = \varphi \circ d$ which on $\Omega^0$ is $[\delta_\varphi, d]$. Then $i_\eta = \delta_\varphi$ is the interior product with derivation $\eta$. If $m = -1$ this is the classical product of differential forms with a given vector field. Define a Lie derivative with respect to $\eta$

$$L_\eta := [\delta_\varphi, d] = [i_\eta, d].$$

Then

$$[L_\eta, d] = [[i_\eta, d], d] = (-1)^{m-1}d i_\eta d - (-1)^m d i_\eta d = 0.$$
where $\deg \omega$. This is in analogy to the Cartan formula for $\Omega$. For example if

$$\eta = m \in O$$

In case $m = 0$, for any function $f \in O$ let $f \cdot -$ denote the multiplication by the function $f$

$$[\delta f, d] = df - df \wedge \deg, \quad [\delta f, d] = df.$$  

Remark 1.8. To prove identities like $\delta = \delta'$, where $\delta, \delta'$ are $O$-linear derivations on $\Omega$, it is enough to prove it on $dO \subset \Omega^1$. For example, for vector fields there is an identity

$$[\mathcal{L}_{\eta}, t_\xi] = [t_\eta, \mathcal{L}_\zeta] = t_{[\eta, \zeta]}.$$  

The expressions are $O$-linear, so we can check the equalities by evaluating on $df, f \in O$.

For $\omega \in \Omega^p$ we have the formula

$$[\delta_{\omega} \wedge - , d]^2(\omega) = \begin{cases} 0 & m = 1 \\ \frac{1}{2}m d(\omega \wedge \omega) \wedge \omega & \text{if } m \text{ is odd } \neq 1 \\ (m + p) d\omega \wedge \omega & \text{if } m \text{ is even.} \end{cases}$$

For example if $m = 1 \varphi$ is the contact 1-form on $\mathbb{A}^1$, that is $\sum_{i=1}^m \xi_i dx_i$. Then

$$\omega = \mathcal{L}_{\xi_i^1} \omega = d\xi_i^1 \omega.$$  

In case $m = 0$, for any function $f \in O$ let $f \cdot -$ denote the multiplication by the function $f$

$$[\delta f, d] = df - df \wedge \deg, \quad [\delta f, d] = df.$$  

Let $\eta_1, \ldots, \eta_p \in \text{Der}_k(O)$ (vector fields if $O = O(X)$). Then there is a formula

$$[d, t_{\eta_1} \ldots t_{\eta_p}] = \sum_{1 \leq i \leq p} (-1)^{i-1} t_{\eta_1} \ldots \hat{t}_{\eta_i} \ldots t_{\eta_p} \mathcal{L}_{\eta_i} + \sum_{1 \leq i < j \leq p} (-1)^{i+j-1} t_{\eta_i} \ldots \hat{t}_{\eta_i} \ldots \hat{t}_{\eta_j} \ldots t_{\eta_j}.$$  

where $\deg t_{\eta_i} = -1$ for all $i = 1, \ldots, p$. Similarly

$$[t_{\eta_1} \ldots t_{\eta_p}, d] = \sum_{1 \leq i \leq p} (-1)^{i-1} \mathcal{L}_{\eta_i} t_{\eta_1} \ldots \hat{t}_{\eta_i} \ldots t_{\eta_p} + \sum_{1 \leq i < j \leq p} (-1)^{i+j} \mathcal{L}_{\eta_i} \ldots \hat{t}_{\eta_i} \ldots \hat{t}_{\eta_j} \ldots t_{\eta_j}.$$  

This is in analogy to the Cartan formula for $\omega \in \Omega^{p-1}$

$$(d\omega)(\eta_1, \ldots, \eta_p) = \sum_{1 \leq i \leq p} (-1)^{i-1} \mathcal{L}_{\eta_i} \omega(\eta_1, \ldots, \hat{\eta}_i, \ldots, \eta_p) + \sum_{1 \leq i < j \leq p} (-1)^{i+j} \omega([\eta_i, \eta_j], \eta_1, \ldots, \hat{\eta}_i, \ldots, \hat{\eta}_j, \ldots, \eta_p).$$
1.5 Koszul-Chevalley complex

Let $m$ be a $g$-module, where $g$ is a Lie $k$-algebra. This means that $[,] : g \otimes_k g \rightarrow g$ satisfies the Jacobi identity, each $g \in g$ acts as an endomorphism of a $k$-module $m$, and the map

$$g \rightarrow gl_k(m) = \text{Lie}(\text{End}_k(m)), \quad g \rightarrow \rho_g - \text{action of } g \text{ on } m$$

is a homomorphism of Lie-$k$-algebras. We have

$$\rho_{[g_1,g_2]} = [\rho_{g_1},\rho_{g_2}]$$

and $gl_k(m)$ has the right $g$-module structure

$$\tilde{\rho}_g(m) := mg,$$

$$mg_1g_2 - mg_2g_1 = (\tilde{\rho}_{g_2}\tilde{\rho}_{g_1} - \tilde{\rho}_{g_1}\tilde{\rho}_{g_2})(m) = [\tilde{\rho}_{g_2}\tilde{\rho}_{g_1}, \tilde{\rho}_g] = m[g_2,g_1].$$

This shows that $\tilde{\rho}_g \rightarrow gl(m)$ is an antihomomorphism of Lie algebras (it corresponds to the fact that the inverse $G \rightarrow G, g \mapsto g^{-1}$ corresponds to $g \mapsto -g$ on $g$).

**Definition 1.9.** Koszul-Chevalley complex of a Lie $k$-algebra $g$ with coefficients in $m$

$$C_\ast(g,m) := m \otimes \Lambda_\ast^g, \quad \partial : C_p(g,m) \rightarrow C_{p+1}(g,m),$$

where

$$\partial(m \otimes g_1 \wedge \cdots \wedge g_p) := \sum_{1 \leq i \leq p} (-1)^{i-1} m \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_p +$$

$$+ \sum_{1 \leq i < j \leq p} (-1)^{i+j-1} m \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge \hat{g}_j \wedge \cdots \wedge g_p.$$

$$C_\ast(g,m) := \text{Alt}_\ast(g \times \cdots \times g, m), \quad \partial : C_{p-1}(g,m) \rightarrow C_p(g,m),$$

where for $\gamma \in \text{Alt}^{p-1}(g \times \cdots \times g,m)$ we define $\delta(\gamma) \in \text{Alt}^p(g \times \cdots \times g,m)$ by

$$\delta(\gamma)(g_1, \ldots, g_p) := \sum_{1 \leq i \leq p} (-1)^{i-1} g_i \gamma(g_1, \ldots, \hat{g}_i, \ldots, g_p) +$$

$$+ \sum_{1 \leq i < j \leq p} (-1)^{i+j-1} \gamma([g_i, g_j], g_1, \ldots, \hat{g}_i, \ldots, \hat{g}_j, \ldots, g_p).$$

In the next definition we use a relative Tor and Ext groups, which are the derived functors in the sense of relative homological algebra ([?], [?]).

**Definition 1.10.** Lie algebra homology and cohomology with coefficients in a $g$-module $m$

$$H_\ast(g,m) := H(C_\ast(g,m), \partial) \simeq \text{Tor}^\ast_{(\text{End}_g)^\ast} (k, m),$$

$$H_\ast(g,m) := H(C_\ast^\ast(g,m), \partial) \simeq \text{Ext}^\ast_{(\text{End}_g)^\ast} (k, m).$$
1.6 A relation between Hochschild and Lie algebra homology

Consider the following situation: $A$ is an associative $k$-algebra with unit, $M$ an $A$-bimodule. Let Lie$(A) = A$ as a $k$-module with commutator bracket $[a, b] := ab - ba$. Let $a \in A$ act on $m \in M$ by $m \mapsto am - ma$. Consider $d_\Delta: A \to A \otimes A^{op}$, $a \mapsto 1 \otimes a^{op} - a \otimes 1$,

$$[d_\Delta a, d_\Delta b] = -\left[1 \otimes a^{op}, b \otimes 1\right] - \left[a \otimes 1, 1 \otimes b^{op}\right] + \left[1 \otimes a^{op}, 1 \otimes b^{op}\right] + \left[a \otimes 1, b \otimes 1\right]$$

(because $A \otimes 1$ and $1 \otimes A^{op}$ commute in $A$)

$$= 1 \otimes [a^{op}, b^{op}] + [a, b] \otimes 1$$

$$= 1 \otimes [b, a]^{op} - [b, a] \otimes 1$$

$$= -d_\Delta [a, b].$$

Universal derivation is an antihomomorphism, so

$$-d_\Delta: \text{Lie}(A) \to \text{Lie}(A \otimes A^{op})$$

is a homomorphism of Lie algebras.

In what follows we will use many arguments based on spectral sequences, and the necessary basics of the theory is presented in appendix (B).

Let $R = \mathcal{U}(\text{Lie}(A))$, $S = A \otimes A^{op}$. Any bimodule $N$ can be viewed as a left $A \otimes A^{op}$-module. The base change spectral sequence takes the form

$$E^2_{pq} = \text{Tor}^A_{p} (\text{Tor}^{\mathcal{U}(\text{Lie}(A))}_q (k, A \otimes A^{op}), N)$$

$$a \cdot (b \otimes c^{op}) = ab \otimes c^{op} - b \otimes a^{op}c^{op} = ab \otimes c^{op} - b \otimes (ca)^{op}$$

Assume that $\mathcal{U}(\text{Lie}(A))$ is flat over $k$. Then

$$\text{Tor}^A_p (\text{Tor}^{\mathcal{U}(\text{Lie}(A))}_q (k, A \otimes A^{op}), N) \simeq \text{Tor}^{\mathcal{U}(\text{Lie}(A))}_q (\text{H}_q (\text{Lie}(A); A \otimes A^{op}), N).$$

In our base change spectral sequence we get an edge homomorphism

$$\text{H}_p (\text{Lie}(A); N) \to \text{Tor}^A_p (\text{H}_0 (\text{Lie}(A); A \otimes A^{op}), N).$$

In general if $g$ is a Lie algebra, and $M$ a $g$-module, then $\text{H}_0 (g; M) = M_g$ - the coinvariants of the $g$-action. Thus we have a map from Lie algebra homology to Hochschild homology

$$\text{H}_p (\text{Lie}(A); N) \to \text{Tor}^A_p (\text{H}_0 (\text{Lie}(A); A \otimes A^{op}), N) = \text{Tor}^A_p (A, N) = \text{H}_p (A; N).$$

When $k$ is of characteristic 0, that map, up to a sign, is induced by inclusion

$$\eta: C_\bullet (\text{Lie}(A); N) \to C_\bullet (A; N)$$

$$n \otimes a_1 \wedge \cdots \wedge a_p \mapsto \sum_{l_1, \ldots, l_p} (-1)^{l_1 - \cdots - l_p} n \otimes a_{l_1} \otimes \cdots \otimes a_{l_p},$$

where on the right hand side we have a sum over all permutations of the set $\{1, \ldots, p\}$, and $l_1 \ldots l_p$ denotes the sign of a permutation.
Proposition 1.11. The map $\eta$ is a map of complexes, that is

$$b\eta = -\eta \partial,$$

where $b$ is the Hochschild boundary, and $\partial$ the boundary of the Koszul-Chevalley complex.

Proof. On the left hand side we have:

$$b\eta(n \otimes a_1 \wedge \cdots \wedge a_p) = \sum_{l_1, \ldots, l_p} (-1)^{l_1-1} n a_{l_1} \otimes \cdots \otimes a_{l_p}$$

$$+ \sum_{1 \leq m \leq p-1} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} n a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p}$$

$$+ \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} a_{l_1} n a_{l_1} \otimes \cdots \otimes a_{l_{p-1}}$$

$$= \sum_{1 \leq i \leq p} (-1)^{i-1} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} n a_{l_1} \otimes a_{l_2} \otimes \cdots \otimes a_{l_p}$$

(because $l_2 \ldots l_p = l_2 \ldots l_p \cdot (-1)^{i-1}$)

$$- \sum_{1 \leq i \leq p} (-1)^{i-1} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} a_{l_1} n a_{l_1} \otimes \cdots \otimes a_{l_{p-1}}$$

(because $l_1 \ldots l_{p-1} = l_1 \ldots l_{p-1} \cdot (-1)^{p-i}$)

$$+ \sum_{1 \leq m \leq p-11} \sum_{1 \leq i \leq j \leq p} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} n a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p}$$

$$+ \sum_{1 \leq m \leq p-11} \sum_{1 \leq i \leq j \leq p} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} n a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p}$$

(because $l_1 \ldots l_p \cdot (-1)^m = l_1 \ldots l_{m-1} l_{m+2} \ldots l_p \cdot (-1)^{(i-1)+(j-1)}$)

$$+ \sum_{1 \leq m \leq p-11} \sum_{1 \leq i \leq j \leq p} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} n a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p}$$

$$+ \sum_{1 \leq m \leq p-11} \sum_{1 \leq i \leq j \leq p} \sum_{l_{1, \ldots, l_p}} (-1)^{l_1-1} n a_{l_1} \otimes \cdots \otimes a_{l_m} a_{l_{m+1}} \otimes \cdots \otimes a_{l_p}$$

$$= \eta \left( \sum_{1 \leq i \leq p} (-1)^i [a_i, n] \otimes a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_p \right)$$

$$+ \sum_{1 \leq i \leq j \leq p} (-1)^{i+j} [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_p \right)$$

$$= -\eta \partial (n \otimes a_1 \wedge \cdots \wedge a_p).$$

$\square$
1.7 Poisson trace

Consider the Lie algebra of derivations Der $\mathcal{O} = \text{Der}_k \mathcal{O}$. The algebra $\mathcal{O}$ is always a Der $\mathcal{O}$-module via the natural representation. Let $\varphi \in \Omega^p_{\mathcal{O}/k}$. Then it defines an alternating $\mathcal{O}$-linear map

$$\varphi: \text{Der} \mathcal{O} \times \cdots \times \text{Der} \mathcal{O} \to \mathcal{O}$$

$$((\eta_1, \ldots, \eta_p) \mapsto \varphi(\eta_1, \ldots, \eta_p) := \iota_{\eta_p} \ldots \iota_{\eta_1} \varphi \in \Omega^0 = \mathcal{O}).$$

There is an $\mathcal{O}$-linear map,

$$\Omega^p \to \text{Alt}^p \mathcal{O}(\text{Der} \mathcal{O}, \mathcal{O}) \hookrightarrow \text{Alt}^p_k(\text{Der} \mathcal{O}, \mathcal{O})$$

which transforms the de Rham differential $d$ into $\delta$

$$d \varphi \mapsto \delta(\iota_{\eta_p} \ldots \iota_{\eta_1} \varphi).$$

(Cartan’s picture of de Rham complex).

Let $\Omega^\text{vol} = \Omega^n$, where $n$ is such that $\Omega^n \neq 0$, $d: \Omega^n \to \Omega^{n+1}$ identically 0. Then

$$C_\bullet(\text{Der} \mathcal{O}; \Omega^\text{vol}) = \Omega^\text{vol} \otimes_k \Lambda^\bullet_k \text{Der}_k \mathcal{O} \to \Omega^\text{vol} \otimes_\mathcal{O} \Lambda^\bullet \text{Der}_k \mathcal{O}$$

where the last epimorphism is $\mathcal{O}$-linearization and is an isomorphism if $\mathcal{O}$ is smooth algebra of dim $n$.

Claim 1.12. The kernel of $\mathcal{O}$-linearization is a subcomplex of $C_\bullet(\text{Der} \mathcal{O}; \Omega^\text{vol})$.

For $\nu \in \Omega^\text{vol} = \Omega^n$

$$\nu \otimes \eta_1 \wedge \cdots \wedge \eta_p \mapsto \iota_{\eta_1} \ldots \iota_{\eta_p} \nu \in \Omega^{n-p} =: \Omega_p.$$  

The composition

$$C_\bullet(\text{Der} \mathcal{O}; \Omega^\text{vol}) \to \Omega^\text{vol} \otimes_\mathcal{O} \Lambda^\bullet \text{Der}_k \mathcal{O}$$

is the map of complexes. It suffices to apply the formula for $[d, \iota_{\eta_1} \ldots \iota_{\eta_p}]$ only to $n$-forms.

$$(C_\bullet(\text{Der}_k \mathcal{O}, \Omega^\text{vol}), \partial) \to (\Omega_\bullet, d)$$

(Spencer’s picture of de Rham complex).

Now we fix the volume form $\nu$, and denote

$$\text{Der}_k \mathcal{O}_\nu := \{\text{derivations annihilating } \nu\}.$$  

There is an $\mathcal{O}$-module morphism

$$\mathcal{O} \to \Omega^\text{vol}, \quad f \mapsto f\nu,$$

$$C_\bullet(\text{Der}_k \mathcal{O}_\nu, \mathcal{O}) \to C_\bullet(\text{Der} \mathcal{O}, \Omega^\text{vol}) \to \Omega_\bullet$$

("Divergentless vector fields").

Suppose that $\mathcal{O} = \mathcal{O}(X)$, where $X$ is a symplectic manifold of dimension $2n$, $\omega \in \Omega^2$ is closed and nondegenerate.

$$\omega: \text{Der} \mathcal{O} \to \Omega^1, \quad \eta \mapsto \iota_{\eta} \omega$$

is injective. Furthermore $\omega^n \in \Omega^\text{vol}$ and we can take $\nu = \omega^n$.  

15
Define \( \text{Ham}(X, \omega) \subset \text{Der}_k \mathcal{O}_\omega \) as

\[
\text{Ham}(X, \omega) := \{ \eta \in \text{Der}_k \mathcal{O}_\omega \mid L_\eta \omega = 0 \}.
\]

Define \( \text{Poiss}(X, \omega) \) as an algebra \( \mathcal{O} \) with the Poisson bracket

\[
\{ f, g \} := \mathcal{L}_{H_f} g = \omega(H_f, H_g) = i_{H_f} i_{H_g} \omega,
\]

where \( H_f \) is the vector field characterized by

\[
i_{H_f} \omega = -df.
\]

There is a homomorphism of Lie algebras

\[
\text{Poiss}(X, \omega) \to \text{Ham}(X, \omega),
\]

and an \( \mathcal{O} \)-linear map of complexes

\[
C_\bullet(\text{Poiss}(X, \omega), \text{ad}) \to C_\bullet(\text{Ham}(X, \omega), \omega^n).
\]

\[
f_0 \otimes f_1 \wedge \cdots \wedge f_p \mapsto f_0 \omega^n \otimes f_1 \wedge \cdots \wedge f_p.
\]

There is also a map

\[
C_\bullet(\text{Ham}(X, \omega), \omega^n) \to \Omega_\bullet,
\]

\[
f_0 \omega^n \otimes f_1 \wedge \cdots \wedge f_p \mapsto f_0 i_{H_{f_1}} \cdots i_{H_{f_p}} \omega^n.
\]

We have

\[
\mathcal{L}_{H_f} = [d, i_{H_f}] \omega = 0.
\]

**Proposition 1.13.** For any \( f, g \in \mathcal{O} \)

\[
H_{f,g} = [H_f, H_g].
\]

**Proof.** It is sufficient to prove the corresponding identity for contractions

\[
i_{[H_f, H_g]} = i_{H_{f,g}}.
\]

We have

\[
i_{[H_f, H_g]} \omega = [\mathcal{L}_{H_f}, i_{H_g}] \omega
\]

\[
= \mathcal{L}_{H_f}(i_{H_g} \omega) - i_{H_g} \mathcal{L}_{H_f} \omega
\]

\[
= -\mathcal{L}_{H_f}(dg)
\]

\[
= -d(\mathcal{L}_{H_f} g)
\]

\[
= -d\{ f, g \}
\]

\[
= -i_{H_{f,g}}.
\]

\( \Box \)
There is a well defined map, called a **Poisson trace**

\[ \text{ptr}_\bullet : (C_\bullet(\text{Poiss}(X, \omega); \text{ad}), \partial) \to (\Omega_\bullet, \delta). \]

Let \( Y \) be a symplectic manifold, \( \dim Y = 2n \), with a symplectic 2-form \( \omega \). Then we have a canonical morphism of chain complexes

\[ \text{ptr} : C_\bullet(\text{Poiss}(Y, \omega); \text{ad}) \to \Omega_\bullet(Y), \]

where \( \Omega_q(Y) = \Omega^{\dim Y - q}(Y) \), given by

\[ f_0 \otimes f_1 \wedge \cdots \wedge f_q \mapsto f_0 \iota_{Hf_1} \cdots \iota_{Hf_q} \omega^n. \]

An important special case is when \( Y \) is a symplectic cone, i.e. \( Y \) is acted upon by \( \mathbb{R}^+ \). Let \( \Xi \) be the corresponding Euler field (the image of \( t \frac{d}{dt} \)). We have \( t^* \omega = t \omega \) or equivalently \( L_\Xi \omega = \omega \).

### 1.7.1 Graded Poisson trace

We consider the graded algebra of functions on \( Y \)

\[ \mathcal{O}_\bullet := \bigoplus_{m \in \mathbb{Z}} \mathcal{O}(m), \]

where

\[ \mathcal{O}(m) := \{ f \in \mathcal{O} \mid L_\Xi f = mf \}. \]

Then the Poisson bracket \{\( , \)\} agrees with the grading in the following way

\[ \{ \mathcal{O}(l), \mathcal{O}(m) \} \subseteq \mathcal{O}(l + m - 1). \]

Let

\[ P_l := \mathcal{O}(l + 1), \quad P_\bullet := \bigoplus_{l \in \mathbb{Z}} P_l \]

be the graded Lie algebra when equipped with the Poisson bracket \{\( , \)\}. The map \( f \mapsto Hf \) is a homomorphism of Lie algebras \( \mathcal{O} \to P = \text{Poiss}(Y, \omega) \), and furthermore

\[ L_\Xi Hf = (\deg(f) - 1) Hf. \]

To check this identity one computes

\[ t_{[\Xi, Hf]} \omega = L_\Xi t_{Hf} \omega = -\deg(f) df + df = (1 - \deg(f)) df = (\deg(f) - 1) Hf \]

because \( t_{Hf} \omega = -df \). Thus there is a **graded Poisson trace**

\[ \text{ptr}_\bullet : C_\bullet(P_\bullet, \text{ad}) \to \Omega_\bullet(Y) \]

\[ \text{ptr}_\bullet : \bigoplus_{k \in \mathbb{Z}} C^{(k)}_\bullet(P_\bullet, \text{ad}) \to \Omega_\bullet(k + n)(Y), \]

where

\[ C^{(k)}_\bullet(P_\bullet, \text{ad}) = (P_\bullet \otimes \Lambda^q P_\bullet)(k + q) \]

and \( \partial \) preserves \( k \). Explicitly we have

\[ L_\Xi (f_0 t_{Hf_1} \cdots t_{Hf_q} \omega^n) = (l_0 + (l_1 - 1) + \ldots + (l_q - 1) + m) f_0 t_{Hf_1} \cdots t_{Hf_q} \omega^n \]

\[ = ((l_0 + \ldots + l_q) + n - q) f_0 t_{Hf_1} \cdots t_{Hf_q} \omega^n, \]

\[ (P_\bullet \otimes \Lambda^q P_\bullet)(l) \to \Omega_q(l - q) \]
1.8 Hochschild homology

Let $C_\bullet(CS(X))$ be the completed Hochschild complex of $CS(X)$. Define

$$C_\bullet^{(m)} := C_\bullet(CS(X))/F_{m-1}C_\bullet(CS(X)),$$

where $F_{m-1}C_\bullet(CS(X))$ is the filtration induced by order. Then

$$C_j = \lim_{m \to -\infty} C_j^{(m)}, \quad j \in \mathbb{N}$$

The complexes $C_\bullet^{(m)}$ inherit filtration from $C_\bullet$

$$\{0\} = F_{m-1}C_\bullet^{(m)} \subset F_mC_\bullet^{(m)} \subset \ldots$$

where

$$F_pC_\bullet^{(m)} := \begin{cases} F_pC_\bullet(CS(X))/F_{m-1}C_\bullet(CS(X)) & \text{for } p \geq m-1, \\ 0 & \text{for } p \leq m-1. \end{cases} \quad (1.4)$$

We have

$$C_j^{(m)} = \lim_{p \to -\infty} F_p^{(m)}, \quad m \in \mathbb{Z}, \quad j \in \mathbb{N}.$$  

Let $HH_\bullet^{(m)}$ denote the homology of $C_\bullet^{(m)}$ and $HH_\bullet$ the homology of $C_\bullet$. Our first objective will be to find $HH_\bullet^{(m)}$.

There is a Milnor short exact sequence

$$0 \to \lim^1 H_{q+1}(C_\bullet^{(m)}) \to HH_q(CS(X)) \to \lim H_q(C_\bullet^{(m)}) \to 0.$$  

If the system $\{H_{q-1}(C_\bullet^{(m)})\}_{m \to -\infty}$ satisfies the Mittag-Leffler condition, then $\lim^1$ vanishes.

Suppose $\{V_\lambda\}$ is an inverse system of sets ($k$-modules). It satisfies Mittag-Leffler condition if for all $\lambda$ the system of subsets $(\text{im}(V_\mu \to V_\lambda))$ for $\mu > \lambda$ stabilizes. The inverse system $\{V_\lambda\}$ can be treated as a sheaf $\tilde{V}$ over the indexing set $\Lambda$ with partial order topology. Then

$$\lim^P\{V_\lambda\} = H^P(\Lambda, \tilde{V}),$$

and in particular $\lim\{V_\lambda\} = \Gamma(\Lambda, \tilde{V})$.

**Theorem 1.14 (Emmanouil).** For $\Lambda = \omega$ - the first infinite ordinal, the inverse system of vector spaces $\{V_\lambda\}$ is Mittag-Leffler if and only if one of the following conditions is satisfied

$$\lim^1\{V_\lambda \otimes_k W\} = 0, \text{ for all vector spaces } W \text{ over } k, \quad (1.5)$$

$$\lim^1\{V_\lambda \otimes_k W\} = 0, \text{ for some infinite dimensional vector space } W \text{ over } k. \quad (1.6)$$

Recall that $T^*_0X = T^*X \setminus X$ and $Y^c$ is the $C^*$-bundle over the cosphere bundle $S^*X$ defined as

$$Y^c := T^*_0X \times_{\mathbb{R}^+} C^*$$

$$\downarrow C^*$$

$$S^*X$$
Consider the eigenspace of the action of the Euler field \( \Xi = \sum_{i=1}^{n} \xi_i \partial \xi_i \) on \( T^*_0 X \)

\[
\Omega^*(T^*_0 X)(m) \subset \Omega_{C^\infty}^*(T^*_0 X)
\]

\[
t^* \eta = t^m \eta
\]

Then

\[
\Omega^{**}(T^*_0 X) := \bigoplus_{m \in \mathbb{Z}} \Omega^*(T^*_0 X)(m)
\]

is a bigraded algebra whose cohomology is naturally isomorphic with \( H^*(Y^\circ) \). We denote it by \( H^*_{dR}(Y^\circ) \).

There is a spectral sequence \( \bigtriangledown_{(m),r} \) converging to \( HH_{(m)} \) which is associated with the filtration (1.4) of \( C^*(m) \). Its complete description is provided in the following proposition.

**Proposition 1.15.** Assume \( m \leq - \dim X = -n \). Then

a) the second term of a spectral sequence \( \bigtriangledown_{(m),r} \) which is associated with the filtration on \( C^*(m) \) which is induced by the order filtration as in (1.4) is given by

\[
\bigtriangledown_{(m),2} \simeq \begin{cases} 
H^{n-p}_{dR}(Y^\circ) & q = n \\
\Omega^{2n-m-q}(n-q)/d\Omega^{2n-1-m-q}(n-q) & p = m \\
0 & \text{otherwise}
\end{cases}
\]

b) the spectral sequence \( \bigtriangledown_{(m),r} \) degenerates at \( \bigtriangledown^2 \)

c) the identification in a) are compatible with the spectral sequence morphisms induced by the canonical spectral sequence projections

\[
C^l_{(m)} \twoheadrightarrow C^m_{(m)}
\]

for \( l \leq m \).
Corollary 1.16. The inverse system of the homology groups \( \{ \text{HH}_p^{(m)} \}_{m \in \mathbb{Z}_{< -n}} \) satisfies Mittag-Leffler condition, in fact
\[
\text{HH}_p^{(l_1, m)} = \text{HH}_p^{(l_2, m)}
\]
for any \( l_1 \leq l_2 \leq m < -n \), where \( \text{HH}_p^{(l, m)} := \text{im}(\text{HH}_p^{(l)} \to \text{HH}_p^{(m)}) \).

Proof. From the proposition (1.15) we obtain a commutative diagrams whose rows are exact.

Consider a spectral sequence with \( \, 'E^0_{p\bullet} \) being the \( p \)-th component of the graded complex \( \text{gr}^F(\text{CS}(X)) \).

Taking homology with respect to the differential \( d^0_{p\bullet} : 'E^0_{p\bullet} \to 'E^0_{p, 1} \), we obtain
\[
'F^1_{pq} = \text{HH}_p^{(p+q)}(\mathcal{O}_*(X))(p),
\]
calculated in terms of differential forms.

If \( \mathcal{O} \) is a smooth algebra, there is a map of complexes
\[
(C_*, b) \to (\Omega^*, 0)
\]
\[
f_0 \otimes \cdots \otimes f_q \to f_0 d f_1 \cdots \wedge d f_q.
\]
But instead of this map we take
\[
f_0 \otimes \cdots \otimes f_q \to (-1)^q q! f_0 \xi_{H_1} \cdots \xi_{H_q} \omega^n.
\]
We can compose the two maps
\[
(C_\bullet(Lie(CS(X))), \partial) \longrightarrow (C_\bullet(CS(X)), b) \longrightarrow (\Omega_{\bullet \bullet}, d).
\]
The first map
\[
\eta: a_0 \otimes a_1 \wedge \cdots \wedge a_q \mapsto \sum_{l_1, \ldots, l_q} (-1)^{l_1 + \cdots + l_q} a_0 \otimes a_{l_1} \otimes \cdots \otimes a_{l_q},
\]
is a map of complexes, while the second one is a map of complexes only if \(d = 0\). But the composition is still a map of complexes.

We identified \(E_2^{(m),1}\) with \(\Omega_2^{2n-p-q}(n-q)\) for \(p \geq m\) and \(d^1\) with \(d_{dR}\).

To demonstrate that the spectral sequence degenerates at \(E_2\) one has to show that the only possibly nontrivial differentials
\[
d^{(m),p-m}_{p_n, p-n}: E^{(m),p-m}_{p_n, p-n} \longrightarrow E^{(m),p-m-1}_{m,n+p-m-1}
\]
all vanish. This is a consequence of the commutativity of the diagram
\[
\begin{array}{ccc}
E^{(m),p-m}_{p_n, p-n} & \longrightarrow & E^{(m),p-m-1}_{m,n+p-m-1} \\
\downarrow & & \downarrow \\
E^{(l),p-m}_{p_n, p-n} & \longrightarrow & E^{(l),p-m-1}_{m,n+p-m-1}
\end{array}
\]
for \(l < m\).

Now \(H_\bullet = HH_\bullet(CS(X))\) is the homology of the projective limit \(\lim C_\bullet^{(m)}\). The projective system \(C^{(m)}\) satisfies Mittag-Leffler condition. The same holds for the projective systems of homology groups \(\{HH_\bullet^{(m)}\}_{m \in \mathbb{Z}_{< n}}\) by corollary (1.16). Hence
\[
HH_j = \lim_m HH_j^{(m)} \simeq H_{dR}^{2n-j}(Y^c),
\]
and we proved the theorem.

**Theorem 1.17.** There is a canonical isomorphism
\[
HH_q(CS(X)) \simeq H_{dR}^{2n-q}(Y^c).
\]

### 1.9 Cyclic homology

We will use the Connes double complex \(B_\bullet(CS(X))\). The maps \(I, B, S\) which involve Hochschild and cyclic homology \(HH_\bullet, HC_\bullet\) are induced by morphism of filtered chain complexes.

\[
C_\bullet(CS(X)) \xrightarrow{I} \text{Tot}(B_\bullet(CS(X))) \xrightarrow{B} \text{Tot}(B_\bullet(CS(X)))[2]
\]

\[
\begin{array}{ccc}
CS(X) \otimes^3 & \xrightarrow{B} & CS(X) \otimes^2 \\
\downarrow & & \downarrow \\
CS(X) \otimes^2 & \xrightarrow{B} & CS(X)
\end{array}
\]

\[
\begin{array}{ccc}
CS(X) \otimes^2 & \xrightarrow{B} & CS(X) \\
\downarrow & & \\
CS(X)
\end{array}
\]

21
The first column is a Hochschild complex $C_{\bullet}(CS(X))$. The rest is the same complex but shifted diagonally by 1, so the total complex is shifted by 2.

Let us put

$$B_{\bullet}^{(m)} := B_{\bullet}/F_{m-1}B_{\bullet},$$

where $F_pB_{kl} := F_pC_{l-k}$. Much as we did before we consider the projective system of quotient complexes

$$\text{Tot } B_{\bullet}^{(m)} = \text{Tot } B_{\bullet}/F_{m-1}B_{\bullet}, \quad m \to -\infty.$$  

Then we have

$$B_{kl}^{(m)} = \lim_{p \to -\infty} F_{pkl}^{(m)}, \quad m \in \mathbb{Z}, \ k, l \geq 0$$

and

$$F_{pkl}^{(m)} := F_pB_{kl}/F_{m-1}B_{kl}.$$  

Let $HC_{\bullet}^{(m)}$ denote the homology of $\text{Tot } B_{\bullet}^{(m)}$, and $HC_{\bullet}$ the homology of $\text{Tot } B_{\bullet}$.

**Proposition 1.18.** Assume that $m \leq 0$ and $q \geq 2n + 1$. Then there exist isomorphisms

$$HC_q^{(m)} \cong \begin{cases} H_{\text{ev}}^{\text{DR}}(Y^c) & q \text{ even} \\ H_{\text{odd}}^{\text{DR}}(Y^c) & q \text{ odd} \end{cases}$$

compatible with the canonical maps $HC_{q'}^{(m')} \to HC_q^{(m)}$ for $m' \leq m$.

In particular, the systems $\{HC_{\bullet}^{(m)}\}_{m \in \mathbb{Z}, 0}$ satisfy for $q \geq 2n + 1$ the Mittag-Leffler condition. This gives us a corollary.

**Corollary 1.19.** There are, for $q \geq 2n + 1$, natural isomorphisms

$$HC_q \cong \lim_{m \to -\infty} HC_q^{(m)} \cong \begin{cases} H_{\text{ev}}^{\text{DR}}(Y^c) & q \text{ even} \\ H_{\text{odd}}^{\text{DR}}(Y^c) & q \text{ odd} \end{cases}$$

This corollary together with a theorem (1.17) imply the following theorem for cyclic homology of an algebra of symbols if $\dim H_{\text{DR}}^\bullet(Y^c) < \infty$.

**Theorem 1.20.** The canonical map

$$I : \HH_{\bullet}(CS(X)) \to HC_{\bullet}(CS(X))$$

is injective. In particular

$$HC_{qr}(CS(X)) = \text{gr}^S HC_q(CS(X)) := S_{qr}/S_{q,r-1}, \quad S_{qr} = \ker S_{1+r}^1 \cap HC_q(CS(X))$$

is canonically isomorphic with

$$H_{\text{DR}}^{2n-q+2r}(Y^c), \quad r = 0, 1, \ldots.$$
With some more work we can prove the theorem without assumption of finite dimension of \( H^{\text{dr}}(Y_c) \). Then one represents \( X \) as a union \( \bigcup_{j \in \mathbb{N}} X_j \) where each \( X_j \) is compact (with smooth or empty boundary) and \( X_j \subset \text{Int} X_{j+1} \). Then the restriction maps \( \text{CS}(X) \rightarrow \text{CS}(X_j) \) induce homomorphisms

\[
\theta: H^*_H(\text{CS}(X)) \rightarrow \hat{H}^*_H := \lim_j H^*_H(\text{CS}(X)), \tag{1.7}
\]

\[
\eta: H^*_C(\text{CS}(X)) \rightarrow \hat{H}^*_C := \lim_j H^*_C(\text{CS}(X)). \tag{1.8}
\]

For each \( q \) there is a commutative diagram

\[
\begin{array}{ccc}
\text{HH}_q(\text{CS}(X)) & \xrightarrow{\theta_q} & \hat{\text{HH}}_q \\
\downarrow \cong & & \downarrow \cong \\
\text{H}^{2n-q}_{\text{dr}} & \rightarrow & \lim_j \text{H}^{2n-q}_{\text{dr}}(Y_c^j)
\end{array}
\]

Notice that also the lower arrow is an isomorphism, since

\[
\Omega^*_j = \lim_j \Omega^*_j,
\]

where \( \Omega_j \) denotes the corresponding graded algebra of functions on \( Y_c^j \). Since both projective systems \( \{\Omega^*_j\} \) and \( \{H^*_C(Y_c^j)\} \) satisfy Mittag-Leffler condition, we have that \( \theta \) in (1.7) is an isomorphism.

The naturality of the Connes exact sequence gives us the commutative diagram

\[
\begin{array}{ccccccc}
\cdots & \rightarrow & \hat{\text{HH}}_q & \rightarrow & \hat{\text{HC}}_q & \rightarrow & G \text{HC}_q-2 & \rightarrow & \hat{\text{HH}}_{q-1} & \rightarrow & \cdots \\
& \theta_q \downarrow \cong & & & & \eta_q \downarrow \cong & & \theta_{q-1} \downarrow \cong & & \cdots \\
\cdots & \rightarrow & B \text{HH}_q & \rightarrow & B \text{HC}_q & \rightarrow & G \text{HC}_q-2 & \rightarrow & B \text{HH}_{q-1} & \rightarrow & \cdots 
\end{array}
\]

with a priori only the lower sequence being exact. The exactness of the upper sequence follows from

\[
\begin{array}{c}
\lim \text{HH}_q(\text{CS}(X_j)) = 0, \text{ for all } q \in \mathbb{N},
\end{array}
\]

which is a consequence of the finite-dimensionality of the groups \( \text{HH}_q(\text{CS}(X_j)) = H^{\text{dr}}_*(Y_c^j) \). Thus the "five lemma" and an easy inductive argument prove that \( \eta \) is an isomorphism and \( B = 0 \).

Now it remains to prove the proposition (1.18). The filtration \( \{F^{(m)}_j \mid p = m, m+1, \ldots\} \) on \( E^{(m)}_* \) induces a filtration on \( \text{Tot} E^{(m)}_* \). Denote by \( E^{(m),r}_{pq} \) the associated spectral sequence which converges to \( \text{HC}^{(m)}_* \).

This spectral sequence is a priori located in the region \( \{(p, q) \mid p \geq m, p + q \geq 0\} \). We
shall see that $E^{(m),r}_{pq}$ for $r \geq 1$ vanishes in fact outside the region shown below

i.e. $E^{(m),r}_{pq} = 0$ also if $p + q \geq 2n$ and $p \neq 0$.

Indeed, $E^{(m),1}_{pq}$ is equal, for $p \geq m$, to

$$H_{p+q}(\text{Tot } B_\bullet(\mathcal{O})(p)) = HC_{p+q}(\mathcal{O})(p).$$

Actually, the first spectral sequence of the double complex $B_\bullet(\mathcal{O})(p)$ degenerates at $E^2$ yielding thus that

$$E^{(m),1}_{pq} \simeq \Omega^{p+q}_\mathcal{O}(p)/d\Omega^{p+q-1}_\mathcal{O}(p), \quad p \geq m, \ p \neq 0,$$

and

$$E^{(m),1}_{0q} \simeq H^{\bar{q}}_{\text{dR}}(Y^c), \quad q \geq 2n,$$

where $\bar{q}$ is the parity of $q$ and $H^{\bullet}_{\text{dR}} = H^{(0)}_{\text{dR}}(Y^c) \oplus H^{(1)}_{\text{dR}}(Y^c)$. This implies the required location of non-vanishing $E^{(m),r}_{pq}$ and as a corollary gives

$$HC_{q}^{(m)} \simeq E^{(m),1}_{0q} \simeq H^{\bar{q}}_{\text{dR}}(Y^c)$$

for $q \geq 2n + 1$. The isomorphisms are also compatible with the canonical mappings $HC_{q}^{(m')} \rightarrow HC_{q}^{(m)}$. 

24
1.9.1 Further analysis of spectral sequence

We will use the notation $E^{(m),r}_{pq}$ for the earlier spectral sequence converging to Hochschild homology $HH^{(m)}$.

First, let us consider the morphism of spectral sequences induced by $S$

$$S^{(m),r}_{pq} : E^{(m),r}_{pq} \rightarrow E^{(m),r}_{p,q-2}$$

For $r=1$ we have

$$E^{(m),1}_{pq} = \begin{cases} HC_{p+q}(O)(p), & O = \text{gr}(CS(X)) = \bigoplus_{p \in \mathbb{Z}} O(p) \\ 0 & p \geq m \\ 0 & p < m \end{cases}$$

Then

$$E^{(m),1}_{pq} : S^{(m),1}_{pq} \rightarrow E^{(m),1}_{p,q-2}$$

is the corresponding component of the $S$-map on cyclic homology of graded algebra $O$.

If $p = 0$

$$HC_{p+q}(O) = \Omega^q \oplus H_{dR}^{q-2} \oplus H_{dR}^{q-4} \oplus \ldots,$$

where

$$\Omega^\bullet := \Omega^\bullet _O, \quad H_{dR}^\bullet := H^\bullet (\Omega^\bullet).$$

$$\overline{\Omega}^k(p) := \Omega^k(p)/d\Omega^{k-1}(p)$$

For $p \neq 0$

$$HC_{p+q}(O)(p) = \begin{cases} \overline{\Omega}^{p+q}(p) & p \geq m \\ 0 & p < m \end{cases}$$

$p = -2$ $p = -1$ $p = 0$ $p = 1$ $p = 2$

$$\begin{array}{ccc}
\overline{\Omega}^{-1}_q(-2) & \xrightarrow{d} & \overline{\Omega}^{-1}_q(-1) \\
0 & \xrightarrow{d} & 0 \\
\overline{\Omega}^{-1}_q(-3) & \xrightarrow{d} & \overline{\Omega}^{-1}_q(-2) \oplus H_{dR}^{q-4} \oplus \ldots & \xrightarrow{d} & \overline{\Omega}^{1+1}_q(1) & \xrightarrow{d} & \overline{\Omega}^{q+2}_q(2) \\
\end{array}$$

where for $p = 0$ we have

$$\begin{array}{ccc}
\overline{\Omega}^q & \oplus & H_{dR}^{q-2} \oplus H_{dR}^{q-4} \oplus \ldots \\
0 & \oplus & \overline{\Omega}^{q-2} \oplus H_{dR}^{q-4} \oplus \ldots
\end{array}$$
Denote
\[ E^{(m),1}_{pq} := \begin{cases} \Omega^{p+q}(\mathcal{O}) & p \geq 0 \\ 0 & p < 0 \end{cases} \]

**Corollary 1.21.** There is an isomorphism of chain complexes
\[ (E^{(m),1}_{\bullet,q}, d_{\bullet,q}) \simeq (E^{(m),1}_{\bullet,q} \oplus (H^{q-2}_{dR} \oplus H^{q-4}_{dR} \oplus \ldots)[0], d) \]
and there is an exact sequence of complexes
\[
\begin{array}{cccccccc}
0 & \to & H^{q-1}_{dR} & \to & (E^{(m),1}_{\bullet,q}) & \to & (E^{(m),1}_{\bullet,q}, d^1) & \to & 0 \\
& & \downarrow & & \downarrow s & & \downarrow & & \\
& & H^{q-1}_{dR} & \to & (E^{(m),1}_{\bullet,q}, d^1) & \to & (E^{(m),1}_{\bullet,q}, d^1) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & & \\
\end{array}
\]

Consider the second spectral sequence of the double complex but arranged according to conventions of Cartan-Eilenberg’s book. Denote it by \( qE^{*}_{\bullet,\bullet} \), although it depends also on \( m \). The \( qE^{*}_{\bullet,\bullet} \) looks as follows.

There is an isomorphism
\[ E^{(m),2}_{pq} \simeq E^{(m),2}_{p+1,q+1} \]
except \((p, q) = (0, q), (1, q - 1), (1, q), (2, q)\).

The term \( E^{(m),2}_{pq} \) appears twice, in \( qE^{*}_{\bullet,\bullet} \) and \( q+1E^{*}_{\bullet,\bullet} \).

There are two cases:

\( q < n \) then for \( l = \left[ \frac{q}{2} \right] + 1 \)
\[ E^{(m),2}_{0} \cong E^{(m),2}_{-1,q-1} \cong E^{(m),2}_{-2,q-2} \cong \ldots \cong E^{(m),2}_{-l,q-1} \cong H_{C_{q-2l}(\mathcal{O})}(l) = 0 \]
because \( q - 2l < 0 \).
The $\mathcal{E}^1$-term is the same as the $\mathcal{E}^2$-term:

$$
\begin{array}{ccccccc}
E_{0,q-1}^{(m),2} & = 0 & E_{1,q-1}^{(m),2} & = 0 & E_{2,q-1}^{(m),2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_{1,q}^{(m),2} & E_{2,q}^{(m),2} & \end{array}
$$

In $\mathcal{E}^3$ there are only two terms and the spectral sequence collapses at $\mathcal{E}^4$.

$$
\begin{array}{ccccccc}
E_{2,q-1}^{(m),2} & \simeq & E_{3,q}^{(m),2} & \simeq & E_{4,q+1}^{(m),2} & \simeq & \cdots & \simeq & E_{2+l,q+l-1}^{(m),2} & \simeq & \Omega^{2l+q-1}(2+l) = 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$q - 1 \geq n$ then for $l = n - \left[\frac{q}{2}\right]$

because $2l + q - 1 > 2n$. 

$$
\begin{array}{ccccccc}
E_{0,n-1}^{(m),2} & \simeq & E_{2,n-1}^{(m),2} & \simeq & E_{3,n-1}^{(m),2} \\
E_{n+3}^{\text{dR}} & H_{dR}^{n+2} & H_{dR}^{n+1} & H_{dR}^n & H_{dR}^{n-1} & H_{dR}^{n-2} & H_{dR}^{n-3} \\
0 & E_{-2,n}^{(m),2} & E_{-1,n}^{(m),2} & 0 & 0 & 0 & 0
\end{array}
$$
\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H^{2n} & 0 & 0 & 0 \\
0 & 0 & H^{2n} & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n + 2 & 0 & 0 & 0 & 0 & H^{2n} \\
n + 1 & 0 & 0 & H^{2n} & 0 & 0 \\
n & 0 & H^{2n} & H^{2n-1} & 0 & 0 \\
n - 1 & 0 & 0 & 0 & 0 & 0 \\
n - 2 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
- n & 0 & 0 & 0 & 0 & 0 \\
- n - 1 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
n - 1 & 0 & 0 & 0 & 0 & 0 \\
n & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
1.9.2 Higher differentials

For \( r = 1, 2, \ldots \) the differentials in the spectral sequence are as follows

\[ \sum_{(p,q) \in R} \dim E^r_{pq} \geq \sum_{(p,q) \in R} \dim E^{r+1}_{pq} \geq \cdots \geq \sum_{(p,q) \in R} \dim E^\infty_{pq}. \]

The equality holds if and only if there is no nontrivial differential originating or leaving \( R \), that is the equality

\[ \sum_{(p,q) \in R} \dim E^r_{pq} = \sum_{(p,q) \in R} \dim E^\infty_{pq} \]

is another way of saying that the spectral sequence in region \( R \) degenerates at \( E^r \).

In our spectral sequence

\[ E^{(m),2}_{pq} \Rightarrow H_{p+q}(\text{Tot} B_{\bullet \bullet}(\text{CS}(X))/F_{m-1} \text{Tot} B_{\bullet \bullet}(\text{CS}(X))) \]

We claim that the only nonvanishing differentials \( d^r_{pq} \) for \( r \geq 2 \) are

\[ d^r_{pq} : E^{(m),p}_{pq} \to E^{(m),2}_{0,p+q-1} \]

which inject \( E^{(m),2}_{pq} = E^{(m),2}_{pq} \simeq H_{dR}^{p-2} \) into \( E^{(m),p}_{0,p+q-1} \).

We can define two regions \( R, R' \) as follows.

Then

\[ \sum_{(p,q) \in R} \dim E^r_{pq} = \sum_{(p,q) \in R} \dim E^\infty_{pq}. \]

Suppose that there is no nontrivial differential originating from \( R' \) or nontrivial differential hitting \( R \) and originating outside. Then

\[ \sum_{(p,q) \in R'} \dim E^r_{pq} - \sum_{(p,q) \in R} \dim E^r_{pq} \geq \sum_{(p,q) \in R'} \dim E^{r+1}_{pq} - \sum_{(p,q) \in R} \dim E^{r+1}_{pq} \]
Equality holds if and only if all \( d^r \) inside \( R \) are zero, and then for all \( r > r_0 \) for some \( r_0 \)

\[
\sum_{(p,q) \in R} \dim E^r_{pq} - \sum_{(p,q) \in R'} \dim E^r_{pq} = \sum_{(p,q) \in R} \dim E^r_{pq} - \sum_{(p,q) \in R'} \dim E^\infty_{pq}.
\]

We can write

\[
\sum_{0 \leq q < n} \dim E^{(m),2}_{0q} - \sum_{0 \leq q < n} \dim E^{(m),\infty}_{0q} = \sum_{p > 0} \dim E^{(m),2}_{pq}.
\]

For \( r \geq 2 \) let us introduce the following statements:

(A) \( r \) The natural maps

\[
E^{(m),r}_{pq} \to E^{(m),r}_{pq} \langle Y^{n-1} \rangle
\]

are isomorphisms for \( p > 0, r \) fixed.

(B) \( r \) The differentials

\[
d^r_{pq} : E^{(m),r}_{pq} \to E^{(m),r}_{0,q+r-1}
\]

are injective.

(C) \( r \) The differentials

\[
d^r_{pq} : E^{(m),r}_{pq} \to E^{(m),r}_{p-r,q+r-1}
\]

are zero for \( p \geq r \).

We prove them by induction on \( r \), simultaneously

\[
(B)_2 \quad (A)_2 \quad (B)_3 \quad (B)_2 \land (C)_2 \implies (A)_3
\]

and so on. Furthermore let us introduce two more sequences of statements:

(D) \( r \) For \( p > m \)

\[
d^r_{pq} = \lim \limits_{j \to \infty} d^r_{pq,j}.
\]

(E) \( r \) For \( p > m \)

\[
E^{(m),r}_{pq} = \lim \limits_{j \to \infty} E^{(m),r}_{pq} \langle Y^j \rangle.
\]

These are also proved by induction on \( r \) in the following way. The \( (E)_r \) implies \( (D)_r \) and \( (E)_r \) and \( (D)_r \) together with the condition that \( \{ E^{(m),r}_{pq} \langle Y^j \rangle \}, \{ E^{(m),r+1}_{pq} \langle Y^j \rangle \} \) satisfy Mittag-Leffler condition, imply \( (E)_{r+1} \).

The \((A)_2\) statement follows from the following remark. Suppose \( H^k_{dR}(Y^c) = 0 \) for \( k > n \) and that \( \dim H^r_{dR}(Y^c) < \infty \). Then

\[
\sum_{j=0}^{2n-2} \dim E^{(m),2}_{0j} - \sum_{p > 0,q} \dim E^{(m),2}_{pq} = \sum_{j=0}^{2n-2} \dim HC_j(CS_Y).
\]

The maps

\[
H^j_{dR}(Y^c) \to H^j_{dR}(\langle Y^k \rangle^c)
\]

are isomorphisms for \( j < k \), monomorphism for \( j = k \), zero for \( j > k + 1 \).
Appendix A

Topological tensor products

Let \((E, \{p_\alpha\}_{\alpha \in A}), (F, \{q_\beta\}_{\beta \in B})\) be vector spaces with the systems of seminorms \(\{p_\alpha\}_{\alpha \in A}, \{q_\beta\}_{\beta \in B}\) respectively. Define a system of seminorms on \(E \otimes F\) by

\[
(p_\alpha \otimes q_\beta)(\tau) := \inf_{i \in I} \sum_{i} p_\alpha(e_i)q_\beta(f_i),
\]

where infimum is taken over all representations \(\tau = \sum_{i} e_i \otimes f_i\) in which \(I\) is a finite set.

**Definition A.1.** A locally convex space \(E \otimes F\) with topology induced by the system of seminorms \(\{p_\alpha \otimes q_\beta\}_{(\alpha, \beta) \in A \times B}\) is called a projective tensor product and denoted by \(E \otimes_\pi F\). Its completion is denoted by \(E \hat{\otimes} \pi F\).

A bilinear map

\[\phi: E \times F \to E \hat{\otimes} \pi F, \quad (e, f) \mapsto e \otimes f,\]

is continuous in both variables and has the following universal property.

**Fact A.2.** For every bilinear jointly continuous mapping \(f: E \times F \to W\) into locally convex space \(W\) there exists unique continuous linear map \(L_\phi: E \hat{\otimes} \pi F \to W\) such that following diagram commutes.

\[
\begin{array}{ccc}
E \times F & \xrightarrow{f} & W \\
\downarrow \phi & & \downarrow \phi \\
E \hat{\otimes} \pi F & \xrightarrow{L_\phi} & W
\end{array}
\]

**Remark A.3.** There are also different tensor products on topological vector spaces, like injective and inductive tensor products, but we will not describe them here.

Suppose that \(E' = \bigcup_{m \in \mathbb{Z}} E'_m\), where

\[
\ldots \subseteq E'_{m-1} \subseteq E'_m \subseteq \ldots
\]

is a \(\mathbb{Z}\)-filtration of \(E'\) by locally convex closed vector subspaces of \(E'\), and analogously for the space \(E''\). Then define

\[
E' \hat{\otimes} E'' := \lim_{(l_1, l_2) \in \mathbb{Z} \times \mathbb{Z}} E'_l \hat{\otimes}_\pi E''_{l_2}.
\]

If for any \(m\) there is a continuous projections \(E'_m \to E'_{m-1}, E''_m \to E''_{m-1}\), then the space \(E'_l \hat{\otimes}_\pi E''_{l_2}\) is a closed subspace in \(E'_m \hat{\otimes}_\pi E''_{m_2}\) for any \(m_1 \geq l_1, m_2 \geq l_2\).
Define a \( \mathbb{Z} \)-filtration on \( E' \hat{\otimes} E'' \)

\[
(E' \hat{\otimes} E'')_m := \bigcup_{(l_1,l_2) \in \mathbb{Z} \times \mathbb{Z}} E'_{l_1} \hat{\otimes} \pi E''_{l_2}.
\]

In similar way we define \( E^{(1)} \hat{\otimes} \ldots \hat{\otimes} E^{(p)} \) with \( \mathbb{Z} \)-filtration

\[
(E^{(1)} \hat{\otimes} \ldots \hat{\otimes} E^{(p)})_m := \bigcup_{(l_1, \ldots, l_p) \in \mathbb{Z}^p} E^{(1)}_{l_1} \hat{\otimes} \ldots \hat{\otimes} \pi E^{(p)}_{l_p}.
\]
Appendix B

Spectral sequences


B.1 Spectral sequence of a filtered complex

Let $(C_*, F, \partial)$ be a filtered chain complex, that is

$$\ldots \subseteq F_p C_* \subseteq F_{p+1} C_* \subseteq \ldots \subseteq C_*.$$  

We say that the filtration is

1. **separable** if $\cap_p F_p C_n = \{0\}$,
2. **complete** if $C_n \cong \lim_p C_n / F_p C_n$,
3. **cocomplete** if $\cup_p F_p C_n \cong C_n$,

for all $n \in \mathbb{Z}$.

We define $E^{0*} := \text{gr}^F C_*$ (the associated graded complex), where $E^{0*} := F_p C_{p+q} / F_{p-1} C_{p+q}$, and $d^{0*}$ is the boundary operator induced by $\partial$, $d^{0*} : E^{0*}_p \to E^{0*}_{p-1}$. Thus $(E^{0*}, d^{0*})$ is the direct sum of complexes

$$(E^{0*}, d^{0*}) = \bigoplus_{p \in \mathbb{Z}} (E^0_{p*}, d^0_{p*}).$$

Next we define

$$E^1_{pq} := H_q(F^0_{p*}, d^0_{p*}) = \frac{\{c \in F_p C_{p+q} \mid \partial c \in F_{p-1} C_{p+q-1}\}}{\{c \in F_p C_{p+q} \mid c = \partial b \text{ for some } b \in F_p C_{p+q+1}\}} \pmod{F_{p-1} C_{p+q}}.$$

On $E^1_{pq}$ the boundary operator $\partial$ induces a boundary operator $d^1_{pq} : E^1_{pq} \to E^1_{p-1,q}$ and so on...
Define for \( r = 1, 2, \ldots \)

\[
E_{pq}^r = \frac{\{ c \in F_p C_{p+q} \mid \partial c \in F_{p-r} C_{p+q-1} \}}{\{ c \in F_p C_{p+q} \mid c = \partial b \, \text{for some} \, b \in F_{p+r-1} C_{p+q+1} \}} \mod F_{p-1} C_{p+q} \\
= Z_{pq}^r + F_{p-1} C_{p+q} \\
\Rightarrow B_{pq}^r + F_{p-1} C_{p+q}.
\]

\[
\cdots \quad F_{p-r} C_{p+q-1} \quad F_{p-r} C_{p+q} \quad F_{p-r} C_{p+q+1} \quad \cdots
\]

\[
\cdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots
\]

\[
\cdots \quad F_{p-1} C_{p+q-1} \quad F_{p-1} C_{p+q} \quad F_{p-1} C_{p+q+1} \quad \cdots
\]

\[
\cdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots
\]

\[
\cdots \quad F_p C_{p+q-1} \quad \partial^0 \quad F_p C_{p+q} \quad \partial^0 \quad F_p C_{p+q+1} \quad \cdots
\]

\[
\cdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots
\]

\[
\cdots \quad F_{p+1} C_{p+q-1} \quad \partial^0 \quad F_{p+1} C_{p+q} \quad \partial^0 \quad F_{p+1} C_{p+q+1} \quad \cdots
\]

\[
\cdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots
\]

\[
\cdots \quad F_{p+r} C_{p+q-1} \quad \partial^0 \quad F_{p+r} C_{p+q} \quad \partial^0 \quad F_{p+r} C_{p+q+1} \quad \cdots
\]

\[
\cdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots
\]

\[
\cdots \quad C_{p+q-1} \quad \partial \quad C_{p+q} \quad \partial \quad C_{p+q+1} \quad \cdots
\]

Now \( E_{pq}^{r+1} \) equipped with the boundary operator induced by \( \partial \) becomes a direct sum of complexes

\[
\cdots \leftarrow E_{p-r,q+r-1}^r \xleftarrow{d_{pq}^r} E_{p+q}^r \xrightarrow{d_{p+r,q-r+1}^r} E_{p-r,q-r+1}^r \rightarrow \cdots,
\]

which we can denote by \((E_{pq}^{r+1}, d_{pq}^r)\). Now \( E_{pq}^{r+1} \) is canonically isomorphic to the homology of the complex \((E_{p+r,q+1}^{r+1}, d_{pq}^r)\) at the \( E_{pq}^r \).
For each \((p, q)\) we defined a system of subobjects of \(F_pC_{p+q}\):
\[
\{0\} = B^0_{pq} \subseteq B^1_{pq} \subseteq \ldots \subseteq B^r_{pq} \subseteq \ldots \\
\subseteq \bigcup_r B^r_{pq} =: B^\infty_{pq} \subseteq Z^\infty_{pq} := \bigcap_r Z^r_{pq} \subseteq \\
\ldots \subseteq Z^1_{pq} \subseteq \ldots \subseteq Z^0_{pq} = F_pC_{p+q},
\]
such that
\[
E^r_{pq} = Z^r_{pq} / B^r_{pq} \mod F_{p-1}C_{p+q}.
\]

Morphism \(\varphi: (C_\bullet, F, \partial) \to (\check{C}_\bullet, F', \partial')\) of filtered complexes induces a morphism
\[
E_{pq}^r(\varphi): E^r_{pq} \to E^r_{pq}, r \geq 0,
\]
of corresponding spectral sequences.

**Theorem B.1 (Eilenberg-Moore).** If \(E_{pq}^r(\varphi)\) is an isomorphism for some \(r\) and both filtrations are complete and cocomplete, then \(\varphi\) is a quasi-isomorphism.

We say that the spectral sequence \(E^\bullet_{pq}\) **converges** to filtered module \(M\) if
\[
E^\infty_{pq} \simeq F_pM_{p+q} / F_{p-1}M_{p+q}, \quad p, q \in \mathbb{Z}.
\]

We write then \(E^r_{pq} \Rightarrow M_{p+q}\).

If the filtration is locally bounded from below (i.e. \(F_pC_n = \{0\}\) for \(p \ll 0\)) and cocomplete, then \(E^\bullet_{pq}\) converges to \(H_*(C_\bullet, \partial)\). The homology of a complex \((C_\bullet, \partial)\) is equipped with canonical filtration
\[
F_pH_*(C_\bullet, \partial) := \text{im}(H_*(F_pC_\bullet, \partial) \to H_*(C_\bullet, \partial)).
\]

We say that the spectral sequence \(E^\bullet_{pq}\) **degenerates** (or **collapses**) at \(E^q\) if \(E^\infty_{pq} \simeq E^\infty_{pq}\).

Consider the \(r\)-th term \(E_r\) of the spectral sequence.

\[
\begin{array}{c}
\bullet \\
\downarrow \\
p \\
\bullet \\
\downarrow \\
q
\end{array}
\]

The source term \(E^r_{pq}\) is mapped to the rightmost one \(E^r_{p'q'}\). There is a sequence of maps
\[
E^r_{pq} \to E^r_{pq} \to \ldots \to E^\infty_{pq} \to H_{p+q}(C),
\]
and similarly
\[
H_{p+q}(C) \to E^\infty_{pq} \to \ldots \to E^r_{p'q'} \to E^r_{p'q'}.
\]

These maps are called the **edge homomorphisms**. For the first quadrant spectral sequence they correspond to maps from leftmost column \(p = 0\)
\[
E^r_{0q} \to H_q(C),
\]
and to bottom row \(q = 0\)
\[
H_p(C) \to E^r_{p0}.
\]
B.2 Examples

Example B.2. Two spectral sequences associated with the double complex \((C_{\bullet \bullet}, \partial', \partial'')\).

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & & \cdots \\
\downarrow & & & & \downarrow \\
\cdots & C_{p-1,q+1} & \xleftarrow{\partial'} & C_{p,q+1} & \xleftarrow{\partial'} & C_{p+1,q+1} & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & C_{p-1,q} & \xleftarrow{\partial'} & C_{p,q} & \xleftarrow{\partial'} & C_{p+1,q} & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & C_{p-1,q-1} & \xleftarrow{\partial'} & C_{p,q-1} & \xleftarrow{\partial'} & C_{p+1,q-1} & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Recall that
\[
\partial'^2 = \partial''^2 = 0, \quad [\partial', \partial''] = \partial' \partial'' + \partial'' \partial' = 0,
\]
and the total complex is defined by

\[
(Tot \, C)_n := \prod_{p=\infty}^{1} C_{p,n-p} \oplus \bigoplus C_{p,n-p}, \quad \partial := \partial' + \partial''.
\]

There are two filtrations on Tot C:

1. filtration by columns

\[
'F_p (Tot \, C)_n := \prod_{r \leq p} C_{r,n-r}
\]
2. Filtration by rows

\[ F_p(\text{Tot } C)_n := \bigoplus_{p \leq s} C_{n-s,s} \]

Filtration by rows is complete and cocomplete only if for all \( n \in \mathbb{Z} \) \( C_{pq} \neq 0 \) for only finite number of \( p, q \) such that \( p + q = n \). Filtration by columns is always complete and cocomplete.

There are two spectral sequences associated to double complex \((C_{\bullet, \bullet}, \partial', \partial'')\).

1. First spectral sequence associated to the filtration by columns

\[ \E^1_{pq} = H_q(C_{p, \bullet}, \partial''). \]

It converges to \( H_{p+q}(C_{\bullet, \bullet}) := H_{p+q}(\text{Tot}(C_{\bullet, \bullet})) \) if \( C_{p,n-p} = 0 \) for \( p \ll 0 \) \((n \in \mathbb{Z})\).

2. Second spectral sequence associated to the filtration by rows

\[ \E^1_{pq} = H_q(C_{\bullet, p}, \partial'). \]

It converges to \( H_{p+q}(C_{\bullet, \bullet}) \) if \( C_{p,n-p} = 0 \) for \( p \ll 0 \) and \( p \gg 0 \) \((n \in \mathbb{Z})\).

Example B.3. Double complex \( B(A)_{\bullet, \bullet} \) (Connes double complex). Let \( A \) be the associative algebra with unit.

\[ B(A)_{pq} := \begin{cases} A^\otimes (q-p+1) & \text{if } q \geq p \geq 0, \\ 0 & \text{otherwise.} \end{cases} \]
Here \( b \) is the Hochschild boundary operator and \( B \) is defined as

\[
B := (1 - t)sN,
\]

where

\[
\begin{align*}
    s(a_0 \otimes \cdots \otimes a_n) &:= 1 \otimes a_0 \otimes \cdots \otimes a_n, \\
    t(a_0 \otimes \cdots \otimes a_n) &:= (-1)^n \otimes a_0 \otimes \cdots \otimes a_{n-1}, \\
    N(a_0 \otimes \cdots \otimes a_n) &:= (id + t + \ldots + t^n)(a_0 \otimes \cdots \otimes a_n).
\end{align*}
\]

**Example** B.4. Double complex \( D(A)_{\bullet\bullet} \). Here \( A \) is commutative \( k \)-algebra with unit.

\[
D(A)_{pq} := \begin{cases} 
    \Omega^{q-p}_{A/k} & \text{if } q \geq p \geq 0, \\
    0 & \text{otherwise.}
\end{cases}
\]

If \( A \cong A \otimes \mathbb{Z} \mathbb{Q} \) (i.e. the additive group \((A,+)\) is uniquely divisible), then the formula

\[
\mu(a_0 \otimes \cdots \otimes a_n) := \frac{1}{n!} a_0^{} da_0 \wedge \cdots \wedge da_n
\]

induces a morphism of double complexes \( \mu : B(A)_{\bullet\bullet} \to D(A)_{\bullet\bullet} \).

On the level of spectral sequences associated with the filtration by columns we obtain surjective maps

\[
E^1(pq)(\mu) : A^{\otimes (q-p+1)} \to \Omega^{q-p}_{A/k}.
\]

These maps are isomorphisms if \( A \) is a function algebra on the smooth algebraic variety over a perfect field (i.e. of characteristic 0 or such that \( k^p = k \) if \( \text{char}(k) = p \)), or inductive limit of such (for example \( A = \mathbb{C} \) as \( \mathbb{Q} \)-algebra).
The first spectral sequence of a double complex \((D(A)_{\bullet\bullet}, 0, d) = \bigoplus_{q \geq 0} (\Omega^q_{A/k} \xrightarrow{d} \cdots \xrightarrow{d} A)\) degenerates at the term \(E^2\):

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\Omega^2_{A/k}/d\Omega^1_{A/k} & \xrightarrow{d} & H^1_{dR}(A) & \xrightarrow{d} H^0_{dR}(A) \\
0 & & 0 & \\
\Omega^1_{A/k}/dA & \xrightarrow{d} & H^0_{dR}(A) & \\
0 & & A & \\
\end{array}
\]

Thus the first spectral sequence of the double complex \((B(A)_{\bullet\bullet}, b, B)\) also degenerates at the term \(E^2\), and we get an isomorphism

\[\text{HC}_n(A) := \text{H}_n(B(A)_{\bullet\bullet}) = \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \oplus H^1_{dR}(A) \oplus H^0_{dR}(A) \oplus \ldots.\]

Example B.5. Let \(P_{\bullet}\) be a projective resolution of a right \(R\)-module \(M\), and \(Q_{\bullet}\) a projective resolution of a left \(R\)-module \(N\). Consider the double complex \(P_{\bullet} \otimes_R Q_{\bullet}\). Then

\[
\begin{align*}
\ell E^2_{pq} &= \begin{cases}
H_p(P_{\bullet} \otimes_R N) & q = 0, \\
0 & q \neq 0
\end{cases} \\

\ell' E^2_{pq} &= \begin{cases}
H_p(M \otimes_R Q_{\bullet}) & q = 0, \\
0 & q \neq 0
\end{cases}
\end{align*}
\]

Both spectral sequences converge to \(H_{p+q}(P_{\bullet} \otimes_R Q_{\bullet}) =: \text{Tor}^R_{p+q}(M, N)\), so we get an important canonical isomorphisms

\[H_p(P_{\bullet} \otimes_R N) \simeq \text{Tor}^R_{p}(M, N) \simeq H_p(M \otimes_R Q_{\bullet}).\]

They express the fact that the bifunctor \(\otimes_R : \textbf{Mod} - R \times R - \textbf{Mod} \rightarrow \textbf{Ab}\) is balanced.

Example B.6. Two hiperhomology spectral sequences. A Cartan-Eilenberg resolution of a complex \((C_{\bullet}, \partial)\) is a double complex \((P_{\bullet\bullet}, \partial', \partial'')\) with augmentation \(\eta : P_{\bullet0} \rightarrow C_{\bullet}\) satisfying the following conditions:

1. for all \(p, q\) the modules \(P_{pq}, \text{im} \partial'_{pq}, \text{ker} \partial'_{pq}, H_p(P_{\bullet q}, \partial')\) are projective,

2. the augmented complexes

\[
\begin{array}{cccc}
P_{\bullet\bullet} & \text{im} \partial'_{\bullet\bullet} & \text{ker} \partial'_{\bullet\bullet} & H_p(P_{\bullet q}, \partial') \\
\eta & \eta & \eta & \eta \\
C_{\bullet} & \text{im} \partial_{\bullet} & \text{ker} \partial_{\bullet} & H_p(C_{\bullet}, \partial) \\
\end{array}
\]

40
are projective resolutions.

\[
\cdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots \\
\cdots \quad P_{p-1,q} \quad \xi_p' \quad P_{p,q} \quad \xi_{p+1} \quad P_{p+1,q} \quad \cdots \\
\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots \quad P_{p-1,q-1} \quad \xi_p' \quad P_{p,q-1} \quad \xi_{p+1} \quad P_{p+1,q-1} \quad \cdots \\
\cdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \cdots \\
\cdots \quad P_{p-1,1} \quad \xi_p' \quad P_{p,1} \quad \xi_{p+1} \quad P_{p+1,1} \quad \cdots \\
\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots \quad P_{p-1,0} \quad \xi_p' \quad P_{p,0} \quad \xi_{p+1} \quad P_{p+1,0} \quad \cdots \\
\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots \quad C_{p-1} \quad \xi_p \quad C_p \quad \xi_{p+1} \quad C_{p+1} \quad \cdots 
\]

Such resolution can be obtained from the arbitrary projective resolutions of \( H_p(C_\bullet, \partial) \) and \( \text{im} \partial_{p-1} \) by gluing them.

\[
\begin{array}{c}
p_H^{pj} \leftarrow \cdots \leftarrow P_{p-1}^{pj} \leftarrow P_{p}^{pj} \leftarrow P_{p+1}^{pj} \leftarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H_p(C_p, \partial) \leftarrow \ker \partial_p \leftarrow \text{im} \partial_{p-1} \leftarrow \text{im} \partial_p \leftarrow C_p \leftarrow \ker \partial_p
\end{array}
\]

For an additive functor \( F \) the hiperhomology spectral sequences are the first and second spectral sequences of a double complex \((F(F_\bullet), F(\partial'), F(\partial''))\)

\[
\begin{aligned}
^1E^1_{pq} &= (L_q F)(C_p), \\
^1E^2_{pq} &= F(p_H^{pj}),
\end{aligned}
\]

and

\[
\begin{aligned}
^2E^1_{pq} &= H_p((L_q F)(C_\bullet)), \\
^2E^2_{pq} &= (L_p F)(H_q(C_\bullet)).
\end{aligned}
\]

Both spectral sequences converge to

\[
\mathbb{L}_{p+q} F(C_\bullet) := H_{p+q}(F(P_\bullet)).
\]

if \( C_\bullet \) is bounded from below, that is \( C_n = 0 \) for \( n \ll 0 \).

Assume that \( C_n = 0 \) for \( n < 0, C_\bullet \) is \( F \)-acyclic, that is \( (L_q F)(C_n) \isomorph C_n, (L_p F)(C_n) = 0 \) for \( p > 0 \), and that

\[
H_n(C_\bullet) = \begin{cases} 
M & n = 0, \\
0 & n > 0.
\end{cases}
\]
Such complex is called an $F$-acyclic resolution of the module $M$. In that case

\[
\begin{align*}
\hat{E}^2_{pq} &\sim \begin{cases} 
H_p(F(C_*)) & q = 0, \\
0 & q \neq 0,
\end{cases} \\
\check{E}^2_{pq} &\sim \begin{cases} 
L_pF(M) & p = 0, \\
0 & p \neq 0.
\end{cases}
\end{align*}
\]

Thus we obtain an isomorphism

\[H_p(F(C_*)) \cong (L_pF)(M).\]

We proved a very important fact, that to compute $(L_pF)(M)$ it is enough to use an arbitrary $F$-acyclic resolution of $M$.

**Example B.7.** Flat module is an $F$-acyclic module for $F = (-) \otimes_R N$, where $N$ is an arbitrary left $R$-module. For $R = \mathbb{Z}$ flat modules are the torsion free abelian groups. Thus

\[0 \leftarrow \mathbb{Q}/\mathbb{Z} \leftarrow \mathbb{Q} \leftarrow \mathbb{Z} \leftarrow 0\]

is a flat resolution of the group $\mathbb{Q}/\mathbb{Z}$ (injective cogenerator of a category of abelian groups $\textbf{Ab}$). From this we obtain

\[\text{Tor}_1^\mathbb{Z}(\mathbb{Q}/\mathbb{Z}, A) = \ker(A \to A \otimes \mathbb{Z} \mathbb{Q}) = \text{Torsion}(A).\]

**Example B.8.** Consider two composable additive functors

\[\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C},\]

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian categories. Let $M$ be an object in $\mathcal{A}$, $P_\bullet$ its projective resolution. In the hyperhomology spectral sequence we put $C_* = G(P_\bullet)$. Then if $G$ sends projective objects into $F$-acyclic objects

\[
\begin{align*}
\hat{E}^2_{pq} &= \text{H}_p((L_qF)(G(P_\bullet))) \cong \begin{cases} 
\text{H}_p((F \circ G)(P_\bullet)) = (L_p(F \circ G))(M) & q = 0, \\
0 & q \neq 0,
\end{cases} \\
\check{E}^2_{pq} &= (L_pF \circ L_qG)(M)
\end{align*}
\]

In this case we obtain that

\[\check{E}^2_{pq} = (L_pF \circ L_qG)(M) \Rightarrow (L_{p+q}(F \circ G))(M).\]

\[
\begin{array}{cccc}
\hat{E}^2_{pq} = E^\infty_{pq} = & \cdots & \cdots & \cdots \\
& 0 & 0 & \cdots & 0 \\
& 0 & 0 & \cdots & 0 \\
(L_0(F \circ G))(M) & (L_1(F \circ G))(M) & \cdots & (L_p(F \circ G))(M)
\end{array}
\]
This spectral sequence is called a spectral sequence of a composition of functors.

Example B.9. Let $\varphi: R \to S$ be a homomorphism of unital rings, $M$ a right $R$-module, $N$ a left $S$-module. Consider a composition

$\text{Mod} - R \xrightarrow{G= (-) \otimes_R S} \text{Mod} - S \xrightarrow{F= (-) \otimes_R N} \text{Ab}$

The spectral sequence of a composition of these two functors (G sends projective $R$-modules into projective $S$-modules) in looks as follows:

$E^2_{pq} = \text{Tor}_{p+q}^S(\text{Tor}_p^R(M, S), N) \Rightarrow \text{Tor}_q^R(M, N)$

and it is called a base change spectral sequence.

Suppose that $R \to S$ is a homomorphism of $k$-algebras, $M_R, S N$ are respectively right $R$-module and left $S$-module. Their tensor product $M \otimes_R N$ gives rise to a sequence of derived functors $\text{Tor}_p^R(M, N)$.

Suppose that $P_\bullet \to M$ is a projective $R$-module resolution of $M$, and $Q_\bullet \to N$ a projective $S$-module resolution for $N$.

$M \otimes_R N \leftarrow P_\bullet \otimes_R Q_\bullet \simeq (P_\bullet \otimes_R S) \otimes_S Q_\bullet$

Suppose $F(\cdot, \cdot)$ is a functor with both covariant arguments.

\[
\begin{align*}
\text{Eq}_{pq}^{2} = \cdots & \hspace{1cm} \cdots \\
(L_0 \circ L_q G)(M) & \hspace{1cm} (L_1 \circ L_q G)(M) & \hspace{1cm} \cdots & \hspace{1cm} (L_p \circ L_q G)(M) \\
(\cdots) & \hspace{1cm} (\cdots) & \hspace{1cm} \cdots & \hspace{1cm} (\cdots) \\
(L_0 \circ L_1 G)(M) & \hspace{1cm} (L_1 \circ L_1 G)(M) & \hspace{1cm} \cdots & \hspace{1cm} (L_p \circ L_1 G)(M) \\
(\cdots) & \hspace{1cm} (\cdots) & \hspace{1cm} \cdots & \hspace{1cm} (\cdots) \\
(L_0 \circ L_0 G)(M) & \hspace{1cm} (L_1 \circ L_0 G)(M) & \hspace{1cm} \cdots & \hspace{1cm} (L_p \circ L_0 G)(M)
\end{align*}
\]
We say that it is **left balanced** if there are isomorphisms $L^{\{1\}}_q \simeq L^{\{2\}}_q \simeq L^{\{2\}}_q$.

\[
\begin{align*}
R^q F(\cdot, \cdot) & \cong R^{q+2}_1 F(\cdot, \cdot) \\
R^{q+2}_1 F(\cdot, \cdot) & \cong R^q_{\{1,2\}} F(\cdot, \cdot) \\
R^q_{\{1\}} F(\cdot, \cdot) & \cong R^q_{\{2\}} F(\cdot, \cdot) \\
R^q_{\{2\}} F(\cdot, \cdot) & \cong R^q_{\{1\}} F(\cdot, \cdot)
\end{align*}
\]

We say that it is **right balanced** if there are isomorphisms $R^{q+2}_{\{1\}} \simeq R^{q+2}_{\{2\}} \simeq R^q_{\{2\}}$.

There is an isomorphism

\[
P \otimes_R N \cong P \otimes_R Q \cong (P \otimes_R S) \otimes_S Q
\]

Taking homology we get

\[
H_p(\text{Tor}_q^R(M, S \otimes_S Q)) \cong \text{Tor}_p^S(\text{Tor}_q^R(M, S), N),
\]

and a base change spectral sequence

\[
E^2_{pq} = \text{Tor}_p^S(\text{Tor}_q^R(M, S), N) \implies \text{Tor}_p^R(M, N).
\]

The boundary maps (transgressions) of this spectral sequences are as follows:

\[
E^2_{0n} = \text{Tor}_n^R(M, S) \otimes_S N \to \text{Tor}_n^R(M, N)
\]

\[
\text{Tor}_n^R(M, N) \to E^2_{0n} = \text{Tor}_n^S(M \otimes_S N)
\]

**Example B.10.** For an unital $k$-algebra $A$ let $\text{Lie}(A)$ denote the associated Lie algebra with bracket $[a, a'] = aa' - a'a$. The universal derivation

\[
d_\Delta : A \to A \otimes_k A^{op}, \quad d_\Delta(a) = 1 \otimes a^{op} - a \otimes 1
\]

is a homomorphism of Lie algebras $\text{Lie}(A) \to \text{Lie}(A \otimes_k A^{op})$, so it induces a homomorphism of associative algebras $R := U(\text{Lie}(A)) \to A \otimes_k A^{op} := S$. Let $M = k$ (trivial representation of a Lie algebra $\text{Lie}(A)$). The base change spectral sequence has the form

\[
E^2_{pq} = \text{Tor}_{p+q}^{U(\text{Lie}(A))}(k, A \otimes_k A^{op}), N) \implies \text{Tor}_{p+q}^{U(\text{Lie}(A))}(k, N),
\]

that is if $A$ is flat over $k$ then

\[
E^2_{pq} = \text{Tor}_{p+q}^{A \otimes_k A^{op}}(H_{q \text{Lie}}^1(A; A \otimes_k A^{op}), N) \implies \text{Tor}_{p+q}^{U(\text{Lie}(A))}(k, N).
\]

Because $k \otimes_{U(\text{Lie}(A))} (A \otimes A^{op}) \simeq A$ as a right $A \otimes A^{op}$-module, we have that the second boundary map gives a canonical homomorphism

\[
H_{q \text{Lie}}^1(A; N) \to H_n(A; N) \simeq E^2_{0n}.
\]
There is a homomorphism of standard chain complexes

$$(C_\bullet(Lie(A); N), \partial) \to (C_\bullet(A, N), b)$$

where

$$\partial(n \otimes a_1 \wedge \cdots \wedge a_n) := \sum_{i=1}^{n} (-1)^i (a_i.n - na_i) \otimes a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} n \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n$$

In the special case $N = A$ we obtain canonical homomorphism

$$H^{Lie}_n(A; \text{ad}) \to HH_n(A)$$

**Example B.11.** Hiper-Tor spectral sequences and K"unneth spectral sequence. For a right $R$-module $M$ and a complex of left modules $C_\bullet$ we define

$$\text{Tor}^R_p(M, C_\bullet) := H_n(P_\bullet \otimes_R C_\bullet)$$

where $P_\bullet \to M$ is a projective resolution of $M$. Then the first and second spectral sequence of a bicomplex $P_\bullet \otimes_R C_\bullet$ are as follows:

$$'E^1_{pq} = P_p \otimes_R H_q(C)$$

$$'E^2_{pq} = \text{Tor}^R_p(M, H_q(C)) \Rightarrow \text{Tor}^R_{p+q}(M, C_\bullet)$$

and

$$''E^1_{pq} = \text{Tor}^R_q(M, C_p)$$

$$''E^2_{pq} = H_p(\text{Tor}^R_q(M, C_\bullet)) \simeq \begin{cases} H_p(M \otimes_R C_\bullet) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

where the isomorphism for $E^2_{pq}$ holds if the complexes $\text{Tor}^R_q(M, C_\bullet)$ are acyclic for $q > 0$, for example if $C_n$ are flat. Then we obtain a K"unneth spectral sequence

$$E^2_{pq} = \text{Tor}^R_p(M, H_q(C)) \Rightarrow H_{p+q}(M \otimes_R C_\bullet)$$

if $C_n = 0$ for $n \ll 0$.

**Example B.12.** If a group $G$ acts on semigroup $S$ and its representation $V$, then $G$ acts on Bar-complex $(B_\bullet(S; V), b')$, where $B_q(S; V) = (kS)^{\otimes q} \otimes_k V$, and $b'$ is a standard boundary operator. Then

$$\text{Tor}_{n,G}^l(G, B_\bullet(S; V)) =: H^G_n(S; V)$$

are the equivariant homology of a semigroup $S$ with coefficients in representation $V$. In an analogous way one can define equivariant homology of a Lie algebra, Hochschild homology, singular homology of a topological space etc.