Dirac operators and Spin structures

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Chapter 1

Dirac operators and Spin structures

1.1 The Dirac operator of \( \mathbb{R}^n \)

First we consider \( n \) even. We shall construct matrices

\[ E_1, E_2, \ldots, E_n, \quad n = 2r \]

each \( E_j \) being \( 2^r \times 2^r \) matrix of complex numbers. In fact each entry will be in \( \{0, 1, -1, i, -i\} \).

Properties of \( E_j \)

1. \( E_j^* = -E_j \),
2. each \( E_j \) is block anti-diagonal

\[ E_j = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix} \]

and each block has size \( 2^{r-1} \times 2^{r-1} \),

3. \( E_j^2 = I_{2^r} \),
4. \( E_j E_k + E_k E_j = 0 \) for \( j \neq k \),
5. \( i^r E_1 E_2 \ldots E_n = \begin{bmatrix} I_{2^{r-1}} & 0 \\ 0 & -I_{2^{r-1}} \end{bmatrix} \)

We will proceed by induction on \( n \) even. For \( n = 2 \) we take

\[ E_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \]

Suppose we have \( E_1, E_2, \ldots, E_n \) of size \( 2^r \times 2^r \). Then we put first \( n \) matrices of size \( 2^{r+1} \times 2^{r+1} \) as

\[ \begin{bmatrix} 0 & E_1 \\ E_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & E_2 \\ E_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & E_n \\ E_n & 0 \end{bmatrix} \]

and two additional matrices

\[ \begin{bmatrix} 0 & -I_{2^r} \\ I_{2^r} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & iI_{2^{r-1}} & 0 \\ 0 & 0 & 0 & iI_{2^{r-1}} \\ iI_{2^{r-1}} & 0 & 0 & 0 \\ 0 & iI_{2^{r-1}} & 0 & 0 \end{bmatrix} \]
**Example 1.1.** For $n = 4$ we have

\[
E_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},
\]

\[
E_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},
\]

For $n$ odd, $n = 2r + 1$, we define matrices $E_1, E_2, \ldots, E_r$ satisfying

1. $E_j^* = -E_j$,
2. $E_2^j = I_{2r}$,
3. $E_j E_k + E_k E_j = 0$ for $j \neq k$,
4. $i^{r+1} E_1 E_2 \ldots E_n = I_{2r}$.

First if $n = 1$ we set

\[ E_1 = [-i]. \]

Then for $n = 2r + 1$ we use $2r$ matrices $E_1, E_2, \ldots, E_{n-1}$ as for the even case and as the last one we put

\[ \begin{bmatrix} -iI_{2r-1} & 0 \\ 0 & iI_{2r-1} \end{bmatrix}. \]

From $E_1, E_2, \ldots, E_n$ we obtain:

1. The Dirac operator of $\mathbb{R}^n$ (described above)
2. The Bott generator vector bundle on $S^n$ ($n$ even)
3. The spin representation of Spin$^c(n)$

#### 1.1.1 Dirac operator

Now we can define **Dirac operator of** $\mathbb{R}^n$. For each $n$ we set

\[ D := \sum_{j=1}^n E_j \frac{\partial}{\partial x_j}. \]

**Example 1.2.** For $n = 1$ we have Dirac operator of $\mathbb{R}$

\[ D = -i \frac{\partial}{\partial x}. \]

For $n = 2$

\[ D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \frac{\partial}{\partial x_2}. \]
For $n = 2r$ and $n = 2r + 1$ $D$ is an unbounded operator on the Hilbert space

$$L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n).$$

$D$ is a first order elliptic differential operator on

$$C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n).$$

With this domain $D$ is symmetric (that is $D$ is formally self-adjoint) and $D$ is essentially self-adjoint (that is $D$ has unique self-adjoint extension). For $n$ even

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}$$

where $D_-$ is the formal adjoint of $D_+$. We will describe these notions in a general context. Let $\mathcal{H}$ be Hilbert space. An unbounded operator on $\mathcal{H}$ is a pair $(\mathcal{D}, T)$ such that

1. $\mathcal{D} \subset \mathcal{H}$ is a vector subspace of $\mathcal{H}$,
2. $\mathcal{D}$ is dense in $\mathcal{H}$,
3. $T: \mathcal{D} \to \mathcal{H}$ is a $\mathbb{C}$-linear map,
4. $(\mathcal{D}, T)$ is closeable, i.e. the closure of graph$(T)$ in $\mathcal{H} \oplus \mathcal{H}$ is the graph of a $\mathbb{C}$-linear map

$$P(\text{graph}(T)) \to \mathcal{H}$$

$$P(u, v) = u.$$

An unbounded operator $(\mathcal{D}, T)$ is symmetric if and only if

$$\langle Tu, v \rangle = \langle u, Tv \rangle \quad \forall \, u, v \in \mathcal{D}.$$

For an unbounded operator $(\mathcal{D}, T)$ on $\mathcal{H}$ let

$$\mathcal{D}(T^*) := \{ u \in \mathcal{H} \mid v \mapsto \langle u, Tv \rangle \text{ extends from } \mathcal{D} \text{ to } \mathcal{H} \text{ extends to be a bounded linear functional on } \mathcal{H} \}$$

For $u \in \mathcal{D}(T^*)$ and $v \in \mathcal{H}$ there exists

$$T^*: \mathcal{D}(T^*) \to \mathcal{H}$$

such that

$$\langle u, Tv \rangle = \langle T^*u, v \rangle.$$

Now $(\mathcal{D}, T)$ is self-adjoint if and only if

$$(\mathcal{D}, T) = (\mathcal{D}(T^*), T^*).$$

Remark 1.3. Symmetric operator needs not to be self-adjoint, but a self-adjoint operator is symmetric.
Example 1.4. Take $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ and
\[
\mathcal{D} = \{ u \in L^2(\mathbb{R}) \mid -i \frac{du}{dx} \in L^2(\mathbb{R}) \text{ in the distribution sense} \}
\]
= \{ u \in L^2(\mathbb{R}) \mid x \hat{u} \in L^2(\mathbb{R}) \},
where $\hat{u}$ is the Fourier transform of $u$ and
\[
x: \mathbb{R} \to \mathbb{R}, \quad x(t) = t, \quad \forall t \in \mathbb{R}.
\]
Then $(C_c^\infty(\mathbb{R}), -i \frac{d}{dx})$ has unique self-adjoint extension $(\mathcal{D}, -i \frac{d}{dx})$.

Let $D$ be Dirac operator of $\mathbb{R}^n$, $n = 2r$ or $2r + 1$.
\[
\Omega^1(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \}
\]
= \{ $f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n \mid f_j: \mathbb{R}^n \to \mathbb{C}, j = 1, 2, \ldots, n$ \}
\[
\Omega^1(\mathbb{R}^n) \text{ acts on } C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n)
\]
in the following way. Let
\[
\omega = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n,
\]
\[
s = (s_1, s_2, \ldots, s_{2r}), \quad s_l: \mathbb{R}^n \to \mathbb{C}, \quad l = 1, 2, \ldots, 2r.
\]
Then
\[
\omega s = \sum_{j=1}^n f_j E_j s.
\]
There is following Leibniz rule for $D$
\[
D(fs) = (df)s + f(Ds),
\]
\[
f: \mathbb{R}^n \to \mathbb{C}, \quad f \in C^\infty(\mathbb{R}^n), \quad df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j.
\]

If $M$ is $C^\infty$-manifold, compact or non-compact, with or without boundary, $\dim M = M$, then the Dirac operator of $M$ is an elliptic operator which is locally like the Dirac operator of $\mathbb{R}^n$.

1.1.2 Bott generator vector bundle

Let $W$ be finite dimensional $\mathbb{C}$-vector space,
\[
T \in \text{Hom}_\mathbb{C}(W, W), \quad T^2 = -I.
\]
Then eigenvalues of $T$ are $\pm i$ and there is decomposition
\[
W = W_i \oplus W_{-i},
\]
\[
W_i = \{ v \in W \mid Tv = iv \}
\]
\[
W_{-i} = \{ v \in W \mid Tv = -iv \}
\]
5
Assume that \( n \) is even, \( S^n \subset \mathbb{R}^{n+1} \)

\[
S^n = \{(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^n \mid a_1^2 + a_2^2 + \ldots + a_{n+1}^2 = 1\}.
\]

We have a map

\[
S^n \to M(2^r, \mathbb{C})
\]

\[(a_1, a_2, \ldots, a_{n+1}) \mapsto a_1 E_1 + a_2 E_2 + \ldots + a_{n+1} E_{n+1} =: F.
\]

From the properties of \( E_j \) we obtain

\[
F^2 = (a_1 E_1 + a_2 E_2 + \ldots + a_{n+1} E_{n+1})^2
\]

\[
= (-a_1^2 - a_2^2 - \ldots - a_{n+1}^2) I
\]

\[= -I
\]

so the eigenvalues of \( F \) are \( \pm i \).

The Bott generator vector bundle \( \beta \) on \( S^n \) is given by

\[
\beta_{(a_1, a_2, \ldots, a_{n+1})} := \text{i-eigenspace of } F
\]

\[
= \{v \in \mathbb{C}^{2^r} \mid F(v) = iv\}
\]

For \( n \) even and \( S^n \subset \mathbb{R}^{n+1} \) there is an isomorphism

\[
K^0(S^n) = \mathbb{Z} \oplus \mathbb{Z}
\]

\[
1 \quad \beta
\]

where 1 = \( S^n \times \mathbb{C} \).

1.2 Spin representation and \( \text{Spin}^c \)

Let \( G \) be a topological group, Hausdorff and paracompact, \( X \) topological space Hausdorff and paracompact. A **principal \( G \)-bundle** on \( X \) is a pair \((P, \pi)\) where

1. \( P \) is a Hausdorff and paracompact topological space with given continuous (right) action of \( G \)

\[
P \times G \to P
\]

\[ (p, g) \mapsto pg \]

2. \( \pi: P \to X \) is a continuous map, mapping \( P \) onto \( X \)

such that given any \( x \in X \), there exists an open subset \( U \) of \( X \) with \( x \in U \) and a homeomorphism

\[
\varphi: U \times G \to \pi^{-1}(U)
\]

with

\[
\pi \varphi(u, g) = u \quad \forall (u, g) \in U \times G
\]

\[
\varphi(u, g_1 g_2) = \varphi(u, g_1) g_2 \quad \forall (u, g_1, g_2) \in U \times G \times G
\]

Such \( \varphi: U \times G \to \pi^{-1}(U) \) is referred to as a local trivialization.
Two principal $G$-bundles $(P, \pi)$ and $(Q, \theta)$ are isomorphic if there exists a $G$-equivariant homeomorphism $f: P \to Q$ with commutativity in the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\ \downarrow \cong & \boxed{} & \downarrow \\ X & \xrightarrow{\theta} & \end{array}
\]

Let $G, H$ be two topological groups and let $(P, \pi)$, $(G, \theta)$ be a principal $G$-bundle and $H$-bundle on $X$. A homomorphism of principal bundles from $(P, \pi)$ to $(Q, \theta)$ is a pair $(\eta, \rho)$ such that

1. $\rho$ is a homomorphism of topological groups $\rho: G \to H$
2. $P \to Q$ is a continuous map with commutativity in the diagrams

\[
\begin{array}{ccc}
P & \xrightarrow{\eta} & Q \\ \downarrow \cong & \boxed{} & \downarrow \\ X & \xrightarrow{\theta} & \end{array} \quad \begin{array}{ccc}
P & \xrightarrow{\eta \times \rho} & Q \times H \\ \downarrow \cong & \boxed{} & \downarrow \\ P & \xrightarrow{\eta} & Q \\ \boxed{} & \boxed{} & \boxed{\pi p = \theta(\eta p), \quad \eta(pg) = (\eta p)(\rho g)}
\end{array}
\]

A homomorphism of principal bundles on $X$ will be denoted $\eta: P \to Q$ and $\rho: G \to H$ will be referred to as homomorphism of topological groups underlying $\eta$.

**Lemma 1.5.** Let $\eta: P \to Q$ be a homomorphism of principal bundles on $X$ with underlying homomorphism of topological groups $\rho: G \to H$. Then for any $x \in X$ there exists an open subset $U$ of $X$ with $x \in U$ and local trivializations

\[
\varphi: U \times G \to \pi^{-1}(U) \\
\psi: U \times H \to \theta^{-1}(U)
\]

such that the diagram

\[
\begin{array}{ccc}
U \times G & \xrightarrow{\varphi} & \pi^{-1}(U) \\ \downarrow \mathrm{Id} \times \eta & \boxed{} & \boxed{\downarrow \eta} \\ U \times H & \xrightarrow{\psi} & \theta^{-1}(U)
\end{array}
\]

commutes.

**Example 1.6.** Let $E$ be $\mathbb{R}$-vector bundle on $X$, $\dim_{\mathbb{R}}(E_p) = n$ for all $p \in X$. Denote

\[
\Delta(E) := \{(p, v_1, v_2, \ldots, v_n) \mid p \in X, v_1, v_2, \ldots, v_n \text{ form a vector space basis for } E_p}\]

$\Delta(E)$ is topologized by

\[
\Delta(E) \subset \underbrace{E \oplus E \oplus \ldots \oplus E}_{n}.
\]

Define an action

\[
\Delta(E) \times \mathrm{GL}(n, \mathbb{R}) \to \Delta(E)
\]
\[(p, v_1, v_2, \ldots, v_n), [a_{ij}] \mapsto (p, w_1, w_2, \ldots, w_n), \]
\[w_j = \sum_{i=1}^{n} a_{ij} v_i, \quad [a_{ij}] \in \text{GL}(n, \mathbb{R})\]

and a map
\[\theta: \Delta(E) \to X,\]
\[\theta(p, v_1, v_2, \ldots, v_n) = p.\]

Then \((\Delta(E), \theta)\) is a principal \(\text{GL}(n, \mathbb{R})\)-bundle on \(X\).

For \(n \geq 3\)
\[\pi_1(\text{SO}(n)) = \mathbb{Z}/2\mathbb{Z}\]
and \(\text{Spin}(n)\) is the unique non-trivial 2-fold cover of \(\text{SO}(n)\). It is a compact connected Lie group.

\[
\begin{array}{c}
\text{Spin}(n) \\
\downarrow \\
\text{SO}(n) \subset \text{GL}(n, \mathbb{R})
\end{array}
\]

There is an exact sequence
\[1 \to \mathbb{Z}/2\mathbb{Z} \to \text{Spin}(n) \to \text{SO}(n) \to 1\]

The group \(\mathbb{Z}/2\mathbb{Z}\) embeds in the \(\text{Spin}(n)\) and \(S^1\) as the \(\{1, -1\}\). We define
\[\text{Spin}^c(n) := S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \text{Spin}(n).\]

Then there is an exact sequence
\[1 \to S^1 \to \text{Spin}^c(n) \to \text{SO}(n) \to 1\]

\(\text{Spin}^c(n)\) is a compact connected Lie group
\[
\begin{array}{c}
\text{Spin}(n) \\
\downarrow \\
\text{Spin}^c(n) \\
\downarrow \\
\text{SO}(n) \subset \text{GL}(n, \mathbb{R})
\end{array}
\]

Example 1.7. For \(n = 1\)
\[\text{Spin}(1) = \mathbb{Z}/2\mathbb{Z}, \quad \text{SO}(1) = 1\]
\[\text{Spin}^c(1) = S^1\]
\[\rho: S^1 \to \text{pt}.\]

For \(n = 2\)
\[\text{Spin}(2) = S^1 = \text{SO}(2)\]
\[\text{Spin}(2) \to \text{SO}(2)\]
\[\zeta \mapsto \zeta^2\]
and
\[\text{Spin}^c(2) = S^1 \times_{\mathbb{Z}/2\mathbb{Z}} \text{Spin}(2)\]
\[\rho(\lambda, \zeta) = \zeta^2.\]
Remark 1.8. Since $\text{SO}(n) \subset \text{GL}(n, \mathbb{R})$ we can view the standard map $\text{Spin}^c(n) \to \text{SO}(n)$ as $\text{Spin}^c(n) \to \text{GL}(n, \mathbb{R})$.

Definition 1.9. A Spin$^c$ datum for an $\mathbb{R}$-vector bundle $E \to X$ is a homomorphism of principal bundles

$$\eta: P \to \Delta(E),$$

where $P$ is a principal Spin$^c(n)$-bundle on $X$ ($n = \dim_{\mathbb{R}}(E_p)$) and the homomorphism of topological groups underlying $\eta$ is the standard map

$$\rho: \text{Spin}^c(n) \to \text{GL}(n, \mathbb{R}).$$

Two Spin$^c$ data $\eta: P \to \Delta(E), \eta': P' \to \Delta(E)$ are isomorphic if there exists an isomorphism $f: P \to P'$ of principal Spin$^c(n)$-bundles on $X$ with commutativity in the diagram

$$\begin{array}{ccc}
P & \xrightarrow{f} & P' \\
\downarrow & & \downarrow \\
\Delta(E) & \xrightarrow{\eta = \eta' \circ f} & \Delta(E)
\end{array}$$

Two Spin$^c$ data $\eta: P \to \Delta(E), \eta': P' \to \Delta(E)$ are homotopic if there exists a principal Spin$^c(n)$-bundle $Q$ on $X$ and a continuous map

$$\Phi: Q \times [0,1] \to \Delta(E)$$

such that

1. For $t \in [0,1]$ each

$$\Phi_t = \Phi(-,t): Q \to \Delta(E)$$

is a Spin$^c$ data.

2. $\Phi_0: Q \to \Delta(E)$ is isomorphic to $\eta: P \to \Delta(E)$

$\Phi_1: Q \to \Delta(E)$ is isomorphic to $\eta': P \to \Delta(E)$

Definition 1.10. A Spin$^c(n)$-structure for $E$ is an equivalence class of Spin$^c(n)$ data, where the equivalence relation is homotopy.

A Spin$^c$ structure for an $\mathbb{R}$-bundle $E$ determines an orientation of $E$. Let $w_1(E), w_2(E), \ldots$ be the Stiefel-Whitney classes of $E$, $w_j(E) H^j(X; \mathbb{Z}/2\mathbb{Z})$-Cech cohomology. Then $E$ is orientable if and only if $w_1(E) = 0$.

A **spin manifold** is a $C^\infty$ manifold $M$, $\dim M = n$, for which the structure group of the tangent bundle $TM$ has been lifted from $\text{GL}(n, \mathbb{R})$ to $\text{Spin}(n)$. Such lifting is possible if and only if

$$w_1(M) = 0, \quad w_1(M) \in H^1(M; \mathbb{Z}/2\mathbb{Z})$$

and

$$w_2(M) = 0, \quad w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z}).$$

A **Spin$^c$ manifold** is a $C^\infty$ manifold $M$, $\dim M = n$, for which the structure group of the tangent bundle $TM$ has been lifted from $\text{GL}(n, \mathbb{R})$ to Spin$^c(n)$. Such lifting is possible if and only if

$$w_1(M) = 0,$$

and

$$w_2(M) \text{ is in the image of } H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}/2\mathbb{Z}).$$
Various well known structures on a manifold $M$ make $M$ into Spin$^c$ manifold

\[(\text{complex analytic}) \Downarrow\]

\[(\text{symplectic}) \xrightarrow{=} (\text{almost complex}) \Downarrow\]

\[(\text{contact}) \xrightarrow{=} (\text{stably almost complex}) \Downarrow\]

\[\text{Spin} \xrightarrow{=} \text{Spin}^c \Downarrow\]

\[(\text{oriented}) \Downarrow\]

A Spin$^c$ manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin$^c$ manifolds. Spin$^c$ structures behave very much like orientations. For example, an orientation on two of three $\mathbb{R}$ vector bundles in a short exact sequence determine an orientation on the third vector bundle. Analogous assertions are true for Spin$^c$ structures.

**Lemma 1.11 (Two out of three lemma).** Let

\[0 \to E' \to E \to E'' \to 0\]

be an exact sequence of $\mathbb{R}$ vector bundles on $X$. If Spin$^c$ structures are given for any two of $E', E, E''$ then a Spin$^c$ structure is determined for the third.

**Corollary 1.12.** If $M$ is a Spin$^c$ manifold with boundary $\partial M$, then $\partial M$ is in canonical way a Spin$^c$ manifold.

**Proof.** There is an exact sequence

\[0 \to T\partial M \to TM|_{\partial M} \to \partial M \times \mathbb{R} \to 0\]

\[\square\]

**Remark 1.13.** If $E$ is orientable ($w_1(E) = 0$), then the set of all possible orientations of $E$ is in 1-1 correspondence with $H^0(X; \mathbb{Z}/2\mathbb{Z})$. If $E$ is Spin$^c$-able ($w_1(E) = 0$ and $w_2(E) \in \text{im}(H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}/2\mathbb{Z}))$), then the set of all possible Spin$^c$-structures for $E$ is then in 1-1 correspondence with $H^0(X; \mathbb{Z}/2\mathbb{Z}) \times H^2(X; \mathbb{Z})$.

1.2.1 Clifford algebras and spinor systems

Let $V$ be a finite dimensional $\mathbb{R}$-vector space, $\langle - , - \rangle$ a positive definite, symmetric, bilinear $\mathbb{R}$-valued inner product on $V$. We can form a tensor algebra

\[TV := \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots\]

with multiplication given by composing the tensors, and then define **Clifford algebra**

\[\text{Cliff}(V) := TV/(v \otimes v + \langle v, v \rangle \cdot 1)\]
where \((v \otimes v + \langle v, v \rangle \cdot 1)\) denotes the two-sided ideal in \(TV\) generated by all elements of the form 
\[
v \otimes v + \langle v, v \rangle \cdot 1, \quad v \in V, \quad 1 \in \mathbb{R}.
\]
As a vector space over \(\mathbb{R}\) \(\text{Cliff}(V)\) is canonically isomorphic to the exterior algebra 
\[
\Lambda^* V = \mathbb{R} \oplus V \oplus \Lambda^2 V \oplus \ldots \Lambda^n V, \quad n = \dim_{\mathbb{R}} V.
\]
Let \(e_1, e_2, \ldots, e_n\) be an orthonormal basis of \(V\). The monomials
\[
e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n}, \quad \epsilon_j \in \{0, 1\}
\]
form a vector space basis of \(\text{Cliff}(V)\). The canonical isomorphism of \(\mathbb{R}\)-vector spaces
\[
\text{Cliff}(V) \to \Lambda^* V
\]
is given by
\[
e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n} \mapsto e_1^{\epsilon_1} \wedge e_2^{\epsilon_2} \wedge \ldots \wedge e_n^{\epsilon_n}.
\]
This isomorphism does not depend on the choice of orthonormal basis of \(V\). 
\[
\dim_{\mathbb{R}}(\text{Cliff}(V)) = 2^n, \quad n = \dim_{\mathbb{R}} V.
\]
In \(\text{Cliff}(V)\) we have following identities
\[
e_j^2 = -1, \quad j = 1, 2, \ldots, n,
\]
\[
e_i e_j + e_j e_i = 0, \quad i \neq j.
\]
We can introduce \(\mathbb{Z}/2\mathbb{Z}\)-grading on \(\text{Cliff}(V)\) in the following way
\[
\text{Cliff}(V) = (\text{Cliff}(V))_0 \oplus (\text{Cliff}(V))_1,
\]
where \((\text{Cliff}(V))_0\) is an \(\mathbb{R}\)-vector space spanned by \(e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n}\) with \(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n\) even, and \((\text{Cliff}(V))_1\) is an \(\mathbb{R}\)-vector space spanned by \(e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n}\) with \(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n\) odd. This \(\mathbb{Z}/2\mathbb{Z}\)-grading does not depend on the choice of orthonormal basis of \(V\).

Take \(\mathbb{R}^n\) with the usual inner product
\[
S^{n-1} \subset \mathbb{R}^n \subset \text{Cliff}(\mathbb{R}^n).
\]
The elements of \(S^{n-1}\) are invertible in \(\text{Cliff}(\mathbb{R}^n)\). Let \(\text{Pin}(n)\) be the subgroup of the invertible elements of \(\text{Cliff}(\mathbb{R}^n)\) generated by \(S^{n-1}\). Then
\[
\text{Spin}(n) = \text{Pin}(n) \cap (\text{Cliff}(\mathbb{R}^n))_0
\]
\[
\rho: \text{Spin}(n) \to \text{SO}(n)
\]
\[
(\rho g)(x) = gxg^{-1}, \quad g \in S^{n-1}, \quad x \in \mathbb{R}^n.
\]
For \(n \geq 3\) this is the unique non-trivial 2-fold covering space of \(\text{SO}(n)\).

Consider complexification
\[
\text{Cliff}_\mathbb{C}(V) := \mathbb{C} \otimes_{\mathbb{R}} \text{Cliff}(V).
\]
Then \(\text{Cliff}_\mathbb{C}(V)\) is a \(C^*\)-algebra with
\[
v^* = -v
\]
for 

\[ v \in V \subset \text{Cliff}(V) \subset \text{Cliff}_\mathbb{C}(V). \]

Let 

\[ \text{Cliff}_\mathbb{C}(\mathbb{R}^n) := \mathbb{C}_\mathbb{R} \text{Cliff}(\mathbb{R}^n), \]

\[ \text{Spin}^c(n) = S^1 \times \mathbb{Z}/2\mathbb{Z} \text{Spin}(n) \subset \text{Cliff}_\mathbb{C}(\mathbb{R}^n). \]

Then \( \text{Spin}^c(n) \) is a subgroup of the group of unitary elements of the \( C^* \)-algebra \( \text{Cliff}_\mathbb{C}(\mathbb{R}^n) \).

Let us now choose an orthogonal basis \( e_1, e_2, \ldots, e_n \) for even-dimensional \( \mathbb{R} \)-vector space \( V \), \( n = 2n = \dim_{\mathbb{R}}(V) \). Recall \( 2^r \times 2^r \) matrices \( E_1, E_2, \ldots, E_n \) defined in the beginning of the chapter and then define a mapping

\[ \text{Cliff}_\mathbb{C}(V) \to M(2^r, \mathbb{C}) \]

\[ e_j \mapsto E_j, \quad j = 1, 2, \ldots, n. \]

This gives an isomorphism of \( C^* \)-algebras \( \text{Cliff}_\mathbb{C}(V) \) and \( M(2^r, \mathbb{C}) \). For an odd dimension \( n = 2r + 1 \) recall \( 2^r \times 2^r \) matrices \( E_1, E_2, \ldots, E_n \) and define two mappings

\[ \varphi_+: \text{Cliff}_\mathbb{C}(V) \to M(2^r, \mathbb{C}) \]

\[ \varphi_+(e_j) = E_j, \quad j = 1, 2, \ldots, n, \]

\[ \varphi_-: \text{Cliff}_\mathbb{C}(V) \to M(2^r, \mathbb{C}) \]

\[ \varphi_-(e_j) = -E_j, \quad j = 1, 2, \ldots, n. \]

Then

\[ \varphi_+ \oplus \varphi_- : \text{Cliff}_\mathbb{C}(V) \to M(2^r, \mathbb{C}) \oplus M(2^r, \mathbb{C}) \]

is an isomorphism of \( C^* \)-algebras.

Remark 1.14. This isomorphisms are non-canonical since they depend on the choice of an orthonormal basis for \( V \).

Let \( E \) be an \( \mathbb{R} \)-vector bundle on \( X \). Assume given an inner product \( \langle -, - \rangle \) for \( E \). Then define \( \text{Cliff}_\mathbb{C}(E) \) as a bundle of \( C^* \)-algebras over \( X \) whose fiber at \( p \in X \) is \( \text{Cliff}_\mathbb{C}(E_p) \).

Definition 1.15. An Hermitian module over \( \text{Cliff}_\mathbb{C}(E) \) is a complex vector bundle \( F \) on \( X \) with a \( \mathbb{C} \)-valued inner product \( \langle -, - \rangle \) and a module structure

\[ \text{Cliff}_\mathbb{C}(E) \otimes F \to F \]

such that

1. \( \langle -, - \rangle \) makes \( F_p \) into a finite dimensional Hilbert space,
2. for each \( p \in X \), the module map

\[ \text{Cliff}_\mathbb{C}(E_p) \to \mathcal{L}(F_p) \]

is a unital homomorphism of \( C^* \)-algebras.

Remark 1.16. Of course all structures here are assumed to be continuous. If \( X \) is a \( C^\infty \) manifold then we could take everything to be \( C^\infty \).
If $E$ is oriented define a section $\omega$ of $\text{Cliff}_C(E)$ as follows. Given $p \in X$, choose a positively oriented orthonormal basis $e_1, e_2, \ldots, e_n$ of $E_p$. For $n$ even, $n = 2r$, set

$$\omega(p) = i^r e_1 e_2 \ldots e_{2r}.$$ 

For $n = 2r + 1$ odd

$$\omega(p) = i^{r+1} e_1 e_2 \ldots e_{2r+1}.$$ 

Then $\omega(p)$ does not depend on the choice of positively oriented orthonormal basis. In $\text{Cliff}_C(E_p)$ we have

$$(\omega(p))^2 = 1.$$ 

If $n$ is odd, then $\omega(p)$ is in the center of $\text{Cliff}_C(E_p)$. Note that to define $\omega$, $E$ must be oriented. Reversing the orientation will change $\omega$ to $-\omega$.

**Definition 1.17.** Let $E$ be an $\mathbb{R}$-vector bundle on $X$. A Spinor system for $E$ is a triple $(\epsilon, \langle -, - \rangle, F)$ such that

1. $\epsilon$ is an orientation of $E$,
2. $\langle -, - \rangle$ is an inner product for $E$,
3. $F$ is an Hermitian module over $\text{Cliff}_C(E)$ with each $F_p$ an irreducible module over $\text{Cliff}_C(E_p)$,
4. if $n = \dim_{\mathbb{R}}(E_p)$ is odd, then $\omega(p)$ acts identically on $F_p$.

**Remark 1.18.** The irreducibility of $F_p$ in (3) is equivalent to $\dim_{\mathbb{C}}(F_p) = 2^r$, where $n = 2r$ or $n = 2r + 1$. In (4) note that $\omega(p)^2 = 1$ so for $n$ odd $\omega(p)$ is in the center of $\text{Cliff}_C(E_p)$. Hence irreducibility of $F_p$ implies that $\omega(p)$ acts either by $I$ or $-I$ on $F_p$. Thus (4) normalizes the matter by requiring that $\omega(p)$ acts as $I$. When $n = \dim_{\mathbb{R}}(E_p)$ is even no such normalization is made.

If $(\epsilon, \langle -, - \rangle, F)$ is a Spinor system for $E$, then $F$ is referred to as the Spinor bundle. Suppose that $n = \dim_{\mathbb{R}}(E_p)$ is even. Let $F_p^+ (F_p^-)$ be the $+1 (-1)$ eigenspace of $\omega(p)$. We have a direct sum decomposition

$$F = F^+ \oplus F^-,$$

where $F^+, F^-$ are $\frac{1}{2} - \text{Spin bundles}$. $F^+ (F^-)$ is a vector bundle of positive (negative) spinors.

Assume we have right and left actions of the group $G$ on topological spaces $X, Y$

$$X \times G \rightarrow X$$

$$G \times Y \rightarrow Y$$

Then

$$X \times_G Y := X \times Y / \sim, \quad (xg, y) \sim (x, gy).$$

**Example 1.19.** Let $E$ be an $\mathbb{R}$-vector bundle on $X$. Then

$$\Delta(E) \times_{\text{GL}(n, \mathbb{R})} \simeq E$$

$$((p, v_1, v_2, \ldots, v_n), (a_1, a_2, \ldots, a_n)) \mapsto a_1 v_1 + a_2 v_2 + \ldots + a_n v_n.$$
Let $E$ be an $\mathbb{R}$-vector bundle on $X$. A Spin$^c$ datum
\[
\eta: P \to \Delta(E)
\]
determines a Spinor system $(\epsilon, \langle -, - \rangle, F)$ for $E$. For $p \in X$, given orientation $\epsilon$, and inner product $\langle -, - \rangle$, an $\mathbb{R}$-basis $v_1, v_2, \ldots, v_n$ of $E_p$ is positively oriented and orthonormal if and only if
\[
(v_1, v_2, \ldots, v_n) \in \text{im}(\eta).
\]
The Spinor bundle for $n = 2r$ or $n = 2r + 1$
\[
F = P \times_{\text{Spin}^c(n)} \mathbb{C}^{2r}.
\]
We have to describe how Spin$^c(n)$ acts on $\mathbb{C}^{2r}$. For $n$ odd Spin$^c(n)$ has an irreducible representation known as its spin representation
\[
\text{Spin}^c(n) \to \text{GL}(2^r, \mathbb{C}), \quad n = 2r + 1.
\]
For $n$ even Spin$^c(n)$ has two irreducible representations known as its $\frac{1}{2}$–Spin representations
\[
\begin{align*}
\text{Spin}^c(n) & \to \text{GL}(2^{r-1}, \mathbb{C}), \\
\text{Spin}^c(n) & \to \text{GL}(2^{r-1}, \mathbb{C}), \quad n = 2r.
\end{align*}
\]
The direct sum
\[
\text{Spin}^c(n) \to \text{GL}(2^{r-1}, \mathbb{C}) \oplus \text{GL}(2^{r-1}, \mathbb{C}) \subset \text{GL}(2^r, \mathbb{C}),
\]
of these representations is the spin representation of Spin$^c(n)$.

Consider $\mathbb{R}^n$ with its usual inner product and usual orthonormal basis $e_1, e_2, \ldots, e_n$
\[
\varphi: \text{Cliff}_\mathbb{C}(\mathbb{R}^n) \to M(2^r, \mathbb{C})
\]
\[
\varphi(e_j) = E_j, \quad j = 1, 2, \ldots, n.
\]
There is a canonical inclusion
\[
\text{Spin}^c(n) \subset \text{Cliff}_\mathbb{C}(\mathbb{R}^n)
\]
and $\varphi$ restricted to Spin$^c(n)$ maps Spin$^c(n)$ to $2^r \times 2^r$ unitary matrices
\[
\text{Spin}^c(n) \to \text{U}(2^r) \subset \text{GL}(n, \mathbb{C}).
\]
This is Spin representation of Spin$^c(n)$ and Spin$^c(n)$ acts on GL$(2^r, \mathbb{C})$ acts on $\mathbb{C}^{2r}$ via this representation.

Let $M$ be $C^\infty$ manifold, possibly $\partial M$ non-empty, $TM$ the tangent bundle of $M$. Then
\[
\begin{align*}
\left( \begin{array}{c}
\text{Spin}^c \text{ datum for } TM \\
\eta: P \to \Delta(TM)
\end{array} \right) \\
\downarrow
\left( \begin{array}{c}
\text{Spinor system for } TM \\
(\epsilon, \langle -, - \rangle, F)
\end{array} \right) \\
\downarrow
\left( \begin{array}{c}
\text{Dirac operator} \\
D: C^\infty_c(M, F) \to C^\infty_c(M, F)
\end{array} \right)
\end{align*}
\]
where $F$ is the Spinor bundle on $M$ and $C^\infty_c(M, F)$ are its $C^\infty$ sections with compact support.

The Dirac operator
\[
D: C^\infty_c(M, F) \to C^\infty_c(M, F)
\]
is such that
1. $D$ is $\mathbb{C}$-linear

\[ D(s_1 + s_2) = Ds_1 + Ds_2, \]

\[ D(\lambda s) = \lambda Ds, \quad s_1, s_2, s \in C_c^\infty(M, F), \quad \lambda \in \mathbb{C}. \]

2. If $f: M \to \mathbb{C}$ is a $C^\infty$ function, then

\[ D(fs) = (df)s + f(Ds). \]

3. If $s_1, s_2 \in C_c^\infty(M, F)$ then

\[ \int_M (Ds_1(x), s_2(x))dx = \int_M (s_1(x), Ds_2(x))dx. \]

4. If $\dim M$ is even, then $D$ is off-diagonal

\[ F = F^+ \oplus F^- \]

\[ D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \]

$D: C_c^\infty(M, F) \to C_c^\infty(M, F)$ is an elliptic first-order differential operator. It can be viewed as an unbounded operator on the Hilbert space $L^2(M, F)$ with the scalar product

\[(s_1, s_2) := \int_M (s_1(x), s_2(x))dx.\]

Moreover it is a symmetric operator.

One proves existence of $D$ by constructing it locally and patching together with a $C^\infty$ partition of unity. The uniqueness of $D$ is obtained by the fact that if $D_0, D_1$ satisfy conditions (1)-(4) above, then

\[ D_0 - D_1: F \to F \]

is a vector bundle map, hence $D_0, D_1$ differ by lower order terms.

**Example 1.20.** Let $n$ be even, $S^n \subset \mathbb{R}^{n+1}$, $D$-Dirac operator of $S^n$, $F$-Spinor bundle of $S^n$, $F = F^+ \oplus F^-$. \n
\[ D: C_c^\infty(S^n, F) \to C_c^\infty(S^n, F) \]

\[ D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \]

\[ D^+: C_c^\infty(S^n, F^+) \to C_c^\infty(S^n, F^-) \]

Then

\[ \text{Index}(D^+) := \dim_\mathbb{C}(\ker D^+) - \dim_\mathbb{C}(\coker D^+). \]

**Theorem 1.21.**

\[ \text{Index}(D^+) = 0. \]

We can tensor $D^+$ with the Bott generator vector bundle $\beta$ from section (1.1.2)

\[ D^+_\beta: C_c^\infty(S^n, F^+ \otimes \beta) \to C_c^\infty(S^n, F^- \otimes \beta). \]

Then we have

**Theorem 1.22.**

\[ \text{Index}(D^+_\beta) = 1. \]