Foliations, C*-algebras and index theory
Part I, II

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Chapter 1

Foliations

1.1 What is a foliation and why is it interesting?

Question 1 (H. Hopf). Is there a completely integrable plane field on $S^3$? (Plane field - two dimensional subbundle $E \subset TS^3$).

Answer 1 (G. Reeb). Yes, it is a tangent bundle to a 2-dimensional Reeb’s foliation of $S^3$, described in the example (1.2(6)).

Question 2 (A. Haefliger). Given a plane subbundle $E$ of $TM$ is it homotopic to an integrable one?

Answer 2 (R. Bott). There exists at least one obstruction; not every subbundle has in its K-theory class an an integrable one.

Roughly speaking, a foliation is the decomposition of a manifold $M^n$ into disjoint family of submanifolds (immersed injectively) of dimension $n - q$, which is locally trivial.

More precisely

Definition 1.1. (1) A codimension $q$ foliation of an manifold $M^n$ is a family $\mathcal{F} = \{L_\alpha\}_{\alpha \in \mathcal{I}}$ of $n - q$-dimensional connected, injectively immersed submanifolds that satisfy

1. $L_\alpha \cap L_\beta \neq \emptyset$ iff $\alpha = \beta$ and $\bigcup_{\alpha \in \mathcal{I}} L_\alpha = M$.

2. For all $p \in M$ there exist open $U \ni p$ and a diffeomorphism

$$\varphi : U \to \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q,$$

such that for all $\alpha \in \mathcal{I}$

$$\varphi((U \cap L_\alpha)\text{conn. comp.}) = \{ x : x_{n-q+1} = c_{n-q+1}, \ldots, x_n = c_n \},$$

where $c_j = \text{constant, } j = n - q + 1, \ldots, n.$

Example 1.2. 1. Fibrations.

2. Surjective submersions.
3. The Kronecker foliation of $\mathbb{T} = S^1 \times S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$.
   Solutions of differential equation $d y = \lambda d x$ with $\lambda = \tan(\theta)$ fixed. If a slope is rational then we get a closed curve - closed leaves of foliation. If $\lambda \notin \mathbb{Q}$ then leaves are dense - they are immersions of $\mathbb{R}$ which is not closed manifold.
   Rough quotient space $M/\mathcal{F}$. Two points are equivalent if and only if they belong to the same leaf. In the Kronecker foliation, when leaves are dense, we get a noncommutative torus.

4. The 1-dimensional Reeb foliation of $\mathbb{T}$.
   ![Picture](image)

5. The 2-dimensional Reeb foliation of a solid torus $D^2 \times S^1$.
   In the universal cover $D^2 \times \mathbb{R} \to D^2 \times S^1$
   ![Picture](image)
   We rotate these curves along vertical axis and define relation $(x, y, z) \sim (x, y, z + 1)$.
   We have one closed leaf (boundary) and rest are open leaves (images of not closed manifolds).

6. The 2-dimensional Reeb foliation of $S^3$.
   
   $$S^3 = D^2 \times S^1 \coprod S^1 \times D^2 / \sim$$
   $$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

   The two tori in above decomposition are
   
   $$\{x \in S^3 \mid x_1^2 + x_2^2 \leq \frac{1}{2}\}$$
   $$\{x \in S^3 \mid x_1^2 + x_2^2 \geq \frac{1}{2}\}$$

   We put on each torus Reeb’s foliation from preceeding example.
   The notion of foliation is interesting for two reasons:
   1. the definition is multifaceted
   2. it gives rise to an interesting equivalence relation on $M$, which in turn gives rise to an interesting quotient “space” $M/\mathcal{F}$.

1.2 Equivalent definitions

**Definition 1.3 (Manifold reformulation).** There exists covering of $M$ by charts $(U_i, \varphi_i)$ such that $\varphi(U_i) = V_i \times W_i$, where $V_i$ and $W_i$ are open subsets of $\mathbb{R}^{n-q}$ and $\mathbb{R}^q$, respectively, with the property that if $U_i \cap U_j \neq \emptyset$ then the diffeomorphism
   $$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$
   is of the form
   $$(x, y) \mapsto (h_{ij}(x, y), g_{ij}(y)), \; g_{ij} : W^o_i \to W^o_j.$$
Definition 1.4 (1-cocycle reformulation). There exists collection $(U_i, f_i, g_{ij})$, where $(U_i)$ is a covering of $M$, $f_i: U_i \to W_i$ are surjective submersions onto open $q$-dimensional manifolds, $g_{ij}: f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ - diffeomorphisms satisfying

$$f_i = g_{ij} \circ f_j$$
onumber
on $U_i \cap U_j$ and $g_{ij} \circ g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$.

Definition 1.5. Let $(M, F)$ be manifold with foliation. The tangent bundle to $F$ is

$$\tau F := \{ X \in TM \mid X \text{ tangent to a leaf} \}.$$ 

Let $S(\tau F)$ denote the space of smooth sections of this bundle. Clearly this is an involutive sub-bundle, i.e.

$$[S(\tau F), S(\tau F)] \subset S(\tau F).$$

because this is local property, obvious on charts.

Conversely by Thm. of Frobenius we can take another

Definition 1.6. Any involutive subbundle $E \subset TM$ is the tangent bundle to a unique foliation.

Equivalently we can say

Definition 1.7. The ideal $I(E)$ generated by the sections of

$$\nu F = \{ \omega \in T^*M \mid \forall X \in \tau F \omega(X) = 0 \}$$

is closed under $d$, i.e. $I(E)$ is a differential ideal.

### 1.3 Holonomy grupoid

Let $x, y \in L \subset M$ be points in a leaf of foliation, $\gamma: [0, 1] \to M$ - path from $x$ to $y$ contained in $L$.

**PICTURE**

Let $W$ -transversal through $x = \varphi^{-1}(x_1 = c_1, \ldots, x_{n-q} = c_{n-q})$. If $x'$ is close to $x$ one can copy $\gamma$ to $\gamma'$, at least for a while. By the compactness of $\gamma$, there exists transversal $T_x \subset W$ such that we reach transversal $T_y$ through $y$, starting from any $x' \in T_x$, and such that $x' \mapsto y' = \gamma'(1)$ is a diffeomorphism $h_\gamma$. We define holonomy of path $\gamma$ as

$$\text{Hol}(\gamma) := \text{germ of } h_\gamma : \text{germ of } T_x \to \text{germ of } T_y$$

Obviously if $\gamma_1 \sim \gamma_2$ are homotopic, then $\text{Hol}(\gamma_1) = \text{Hol}(\gamma_2)$, i.e. holonomy factors through homotopy.

Recall that grupoid is a small category with all arrows invertible.

**Definition 1.8.** Holonomy grupoid

$$\mathcal{G}(F) := \{ (x, \text{Hol}(\gamma), y) \mid \exists \text{ leaf } L \ni x, y, \text{ and path } \gamma: [0, 1] \to L \text{ from } x \text{ to } y \}$$

with objects

$$\mathcal{G}^0 = M$$

and composition

$$(y, \text{Hol}(\delta), z) \circ (x, \text{Hol}(\gamma), y) = (z, \text{Hol}(\delta \circ \gamma), z).$$
Interpretation:

- \((x, \text{Hol(const)}, x)\) “reflexibility” = unit,
- \((x, \text{Hol}(\gamma), y) = (y, \text{Hol}(\gamma^{-1}), x)\) “symmetry” = inverse,
- \((y, \text{Hol}(\delta), z) \circ (x, \text{Hol}(\gamma), y) = (x, \text{Hol}(\delta \circ \gamma), z)\) “transitivity” = composition.

Let \(T\) be a complete transversal to \(\mathcal{F}\) i.e. \(T\) is an immersed submanifold, transverse to each leaf and intersecting each leaf at least once.

\[
\mathcal{G}_T(\mathcal{F}) = \{ (x, \text{Hol}(\gamma), y) \in \mathcal{G}(\mathcal{F}) \mid x, y \in T \}
\]

\[
C^\infty_c(\mathcal{G}_T(\mathcal{F})) \hookrightarrow C^*(\mathcal{G}_T(\mathcal{F}))
\]

\[
(f * g)(\text{Hol}(\gamma)) = \sum_{\text{Hol}(\gamma_1) \text{Hol}(\gamma_2) = \text{Hol}(\gamma)} f(\text{Hol}(\gamma_1))g(\text{Hol}(\gamma_2))
\]

### 1.4 How to handle “\(M/\mathcal{F}\)”

“\(M/\mathcal{F}\)” = grupoid \(\mathcal{G}(\mathcal{F})\)

(A) “Homotopy quotient” approach, or equivalently via classifying spaces. This is similar in spirit to

“\(M/\Gamma\)” \(\hookrightarrow M \times_\Gamma \mathcal{E} \Gamma \rightarrow B \Gamma\),

where \(\Gamma\) is a group.

“\(M/\mathcal{F}\)” \(\sim B \mathcal{G}(\mathcal{F}) \rightarrow B \Gamma_q\)

(B) “Topos” approach, by extending “duality”

Topological spaces \(\leftrightarrow\) Sheaves of sets,

and associating a suitably defined topos to \(\mathcal{G}(\mathcal{F})\).

(C) Connes noncommutative geometry approach, by extending the duality

Topological spaces \(\leftrightarrow\) Commutative C*-algebras,

to include \(C^*(\mathcal{G})\), for \(\mathcal{G}\)-groupoid.

### 1.5 Characteristic classes

All approaches produce cohomology groups for grupoids, equivalent for (A) & (B), and cyclic cohomology \(HC^*\) for (C), as well as characteristic maps. They are all “huge” and not well understood. The ones which are best understood are the “geometric” characteristic classes.

1. Bott’s construction a la Chern-Weil.
2. Gelfand-Fuks realization.
3. Hopf-cyclic cohomological construction.
Chapter 2

Characteristic classes

2.1 Preamble: Chern-Weil construction of Pontryagin ring

Let
\[ E \rightarrow M \]
be a real vector bundle. A connection on \( E \) is a linear operator
\[ \nabla : S(E) \rightarrow S(T^*M \otimes E) = \Omega^1(M) \otimes S(E) \]
satisfying following rule
\[ \nabla(fs) = df \otimes s + f \nabla(s). \]

Then \( \nabla \) extends to a graded \( \Omega(M) \)-module map
\[ \nabla : \Omega^*(M) \otimes S(E) \rightarrow \Omega^*(M) \otimes S(E) = \Omega^*(M, E), \]
by
\[ \nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \nabla(s). \]

The Curvature of \( \nabla \): we can view \( \Omega^*(M, E) \) as a module over \( \Omega^*(M) \) and then for any \( \zeta \in \Omega^*(M, E) \) and any \( \omega \in \Omega^*(M) \) we have
\[ \nabla^2(\omega \zeta) = \nabla(d\omega \zeta + (-1)^{\deg \omega} \omega \nabla(\zeta)) = \]
\[ = (-1)^{\deg + 1} d\omega \nabla(\zeta) + (-1)^{\deg} d\omega \nabla(\zeta) + \omega \nabla^2(\zeta) = \omega \nabla^2(\zeta). \]

It means that \( \nabla^2 \) is a local operator - multiplication by an element of the base ring. It follows that
\[ \nabla^2(\zeta) = R \cdot \zeta, \ R \in \Omega^2(M, \text{End}(E)). \]

We call \( R \) a curvature form.

Explicit expression in terms of covariant derivative:
\[ X \rightarrow \text{vector field}, \ \nabla_X(s) = \nabla s(X) \]
\[ \nabla_X : S(E) \rightarrow S(E). \]

Let \( \{X_i\} \) be basis of TM, i.e. linearly independent vector fields, \( \{\omega^i\} \) - its dual basis of 1-forms. Then
\[ \nabla(s) = \sum_i \omega^i \otimes \nabla_{X_i}(s), \text{ hence} \]
\[ \nabla^2(s) = \sum_i d\omega^i \otimes \nabla X_i(s) - \sum_i \omega^i \nabla(\nabla X_i(s)) = \]
\[ = \sum_i d\omega^i \otimes \nabla X_i(s) - \sum_{i,j} \omega^i \land \omega^j \nabla X_j \nabla X_i s. \]

Where the second sum could be written as
\[ \sum_{i,j} \omega^i \land \omega^j \nabla X_j \nabla X_i s = \sum_{i<j} \omega^i \land \omega^j [\nabla X_j, \nabla X_i] s. \]

Write
\[ d\omega^i = \sum_{j<k} f^i_{jk} \omega^j \land \omega^k, \]
with \( f^i_{jk} = d\omega^i(X_j, X_k) = -\omega^i([X_j, X_k]) \). With that, we can rewrite first sum as
\[ \sum_i d\omega^i \otimes \nabla X_i(s) = -\sum_{j<k} \sum_i \omega^i([X_j, X_k]) \omega^j \land \omega^k \otimes \nabla X_i(s) = \]
\[ = -\sum_{j<k} \omega^j \land \omega^k \otimes \nabla [X_j, X_k](s). \]

We just proved

**Lemma 2.1.**

\[ \nabla^2 s = \sum_{j<k} \omega^j \land \omega^k R_{X_j,X_k}(s) = R \cdot s, \]
where
\[ R_{X,Y} = [\nabla X, \nabla Y] - \nabla [X,Y] \in \text{End}(E), \]
and
\[ R = \sum_{j<k} R_{X_j,X_k} \omega^j \land \omega^k. \]

For any Lie algebra \( g \) of a Lie group \( G \), we denote by \( \mathcal{I}(g) \) set of polynomials on \( g \) which are invariant under adjoint action \( \text{Ad} G \). For
\[ P \in \text{Sym}(g^* \otimes \ldots \otimes g^*) \]

it means that
\[ P(\text{Ad}(g)x_1, \ldots, \text{Ad}(g)x_r) = P(x_1, \ldots, x_r), \]
where
\[ \text{Ad}(g)(a) = gag^{-1}. \]

Let \( \mathfrak{gl}_n(\mathbb{R}) \) be the Lie algebra of \( \text{GL}_n(\mathbb{R}) \). The set \( \mathcal{I}(\mathfrak{gl}_n) \) is in fact ring, and is generated by elements
\[ P_{2k}(A) = P_{2k}(A, \ldots, A) = \text{tr}(A^k). \]

**Theorem 2.2 (Chern-Weil).** Let \( P \in \mathcal{I}(\mathfrak{gl}_n(\mathbb{R})) \) be an invariant polynomial of degree \( k \), \( R \) - curvature of connection \( \nabla \) on real vector bundle \( E \rightarrow M \).

1. Then \( P(R) = P(R, \ldots, R) \in \Omega^{2k}(M) \) is closed and its de Rham cohomology class is independent of the connection.
More precisely, if \( \nabla_0, \nabla_1 \) are two connections, then

\[
P(R_1) - P(R_0) = k \cdot d \int_0^1 P(\alpha, R_t, \ldots, R_t) dt,
\]

where \( \alpha \in \Omega^1(M, \text{End}(E)) \) is the difference \( \alpha = \nabla_1 - \nabla_0 \), and \( R_t \) is the curvature of a connection \( \nabla_t = (1 - t) \nabla_0 + t \nabla_1 \).

**Proof.** It is based on the two lemmas.

**Lemma 2.3.** If \( \deg(P) \) is odd, then \( P(R) = 0 \) for any metric connection.

**Proof.** By hypothesis we have using Euclidean structure \((E, \langle -,- \rangle)\)

\[
X(\langle s,t \rangle) = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle.
\]

This implies

\[
XY(\langle s,t \rangle) = \langle [\nabla_X, \nabla_Y]s, t \rangle + \langle s, [\nabla_X, \nabla_Y]t \rangle = \langle \nabla_{[X,Y]}s, t \rangle + \langle s, \nabla_{[X,Y]}t \rangle.
\]

We can write then

\[
\langle R_{X,Y}s, t \rangle + \langle s, R_{X,Y}t \rangle = 0, \text{ i.e.}
\]

\[
R + R^t = 0, \text{ and } P(R) = P(R^t, \ldots, R^t) = (-1)^k P(R).
\]

**Lemma 2.4.** For \( \omega \in S(M, \text{End}(E)) \) one has

\[
d(\text{tr} \omega) = \text{tr}[\nabla, \omega].
\]

**Proof.** Locally, on a chart \( U \) we have \( \nabla = d + \alpha, \alpha \in \Omega^1(U, \text{End}(E)) \). Hence

\[
[\nabla, \omega] = [d + \alpha, \omega] = d\omega + [\alpha, \omega], \text{ and}
\]

\[
\text{tr}[\nabla, \omega] = \text{tr} d\omega + \text{tr}[\alpha, \omega] = d(\text{tr} \omega).
\]

In particular (Bianchi’s identity)

\[
d \text{tr}(R^k) = \text{tr}[\nabla, R^k] = \text{tr}[\nabla, \nabla^{2k}] = 0.
\]

This gives proof of the first part, because polynomials of the form \( \text{tr}(R^k) \) generate \( \mathcal{I}(\mathfrak{gl}_n(\mathbb{R})) \).

For the second part, note that if \( \nabla_t = (1 - t) \nabla_0 + t \nabla_1 \), we have

\[
\frac{d}{dt}(R_t) = \frac{d}{dt}(\nabla_t^2) = \frac{d}{dt}(\nabla_t) \nabla_t + \nabla_t \frac{d}{dt} \nabla_t = [\nabla_t, \nabla_t] = [\nabla_t, \alpha],
\]

where \( \alpha = \nabla_1 - \nabla_0 \). Now

\[
\frac{d}{dt} \text{tr}(R_t^k) = \text{tr} \left( \frac{d}{dt} R_t^k \right) = k \text{tr} \left( \frac{dR_t}{dt} R_t^{k-1} \right) = k \text{tr} \left( [\nabla_t, \alpha] \nabla_t^{2(k-1)} \right) = k d \text{tr}(\alpha R_t^{k-1}).
\]
2.2 Adapted connection and Bott’s theorem

Let $E \subset TM$ be an involutive subbundle and let $Q = TM/E$ with $\pi : TM \to Q$ be the projection.

**Definition 2.5.** An adapted (or $E$-flat) connection on $Q$ is a connection $\nabla$ such that

$$\nabla_X \pi(Z) = \pi([X, Z]), \forall X \in \mathcal{S}(E).$$

This makes sense, since

$$\nabla_{fX} \pi(Z) = \pi([fX, Z]) = -\pi(Z(f)X) + f\pi([X, Z]) = f\nabla_X \pi(Z),$$

and

$$\nabla_X(f\pi(Z)) = \pi([X, fZ]) = \pi(X(f)Z) + f\pi([X, Z]) = X(f)\pi(Z) + f\nabla_X(\pi(Z)).$$

To construct such a connection, take a decomposition $TM = E \oplus Q$ and set

$$\nabla_X \pi(Z) = \nabla_{X_E} \pi(Z) + \nabla_{X_{E\perp}} \pi(Z) = \pi([X_E, Z]) + \nabla_{X_{E\perp}} \pi(Z)$$

where we take an arbitrary connection on $E_{E\perp}$.

**Lemma 2.6.** For any adapted connection $R_{X,Y} = 0, \forall X, Y \in \mathcal{S}(E)$.

**Proof.**

$$R_{X,Y} \pi(Z) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(\pi(Z)) = \pi([X, [Y, Z]] - [Y, [X, Z]] - [[X, Y], Z]) = 0.$$

**Theorem 2.7 (Bott’s vanishing theorem).** Given $E \subset TM$ which is involutive, we have for $Q = TM/E$, $\dim Q = q$

$$\operatorname{Pont}^{>2q}(Q) = 0.$$

**Proof.** Let

$$P_{2k}(A) := \operatorname{tr}(A^k).$$

Then for

$$R = \sum_{i<j} R_{X_i,X_j} \omega^i \wedge \omega^j$$

we have

$$P_{2k}(R) = \operatorname{tr}(R^k) = \sum \operatorname{tr}(R_{X_i,X_j, \ldots, X_{i2k}, X_{j2k}}) \omega^{i1} \wedge \omega^{j1} \wedge \ldots \wedge \omega^{i2k} \wedge \omega^{j2k}.$$ 

If $k > q$, at least one pair belongs to $E$, otherwise

$$\omega^{i1} \wedge \ldots \wedge \omega^{i2k} = 0.$$

**Remark 2.8.**

$$\operatorname{Pont}(Q) = \operatorname{Pont}(TM \ominus E),$$

hence the above is a restriction of $[E] \in K^0(M)$. 

9
2.3 The Godbillon-Vey class

Let $\mathcal{F}$ be a codimension $q$ foliation of $M^n$, $E = \tau \mathcal{F}$, $Q = TM/E$. First, assume that $\mathcal{F}$ is transversally orientable i.e. $\Lambda^q Q$ has nowhere zero section (giving trivialization $\Lambda^q Q \cong M \times \mathbb{R}$).

**Lemma 2.9.** Let $\Omega$ be nonvanishing section of $\Lambda^q Q$. Then
\[ d\Omega = \alpha \wedge \Omega \]
for some $\alpha \in \Omega^1(M, \text{End}(E))$.

**Proof.** It suffices to prove it locally, then patch by partition of unity.

On a chart $U$, choose a basis $\omega_1, \ldots, \omega_q \in \mathcal{I}(E)$ such that
\[ \Omega = \omega_1 \wedge \ldots \wedge \omega_q, \]
\[ d\omega_i = \sum_{j=1}^{q} \alpha_{ij} \wedge \omega_j \]
Then
\[ d\Omega = \sum_{i=1}^{q} (-1)^i \omega_1 \wedge \ldots \wedge d\omega_i \wedge \ldots \wedge \omega_q = \]
\[ = \sum_{i=1}^{q} (-1)^i \omega_1 \wedge \ldots \wedge \left( \sum_{j=1}^{q} \alpha_{ij} \wedge \omega_j \right) \wedge \ldots \wedge \omega_q \]
Only $\alpha_{ii} \wedge \omega_i$ can contribute to the sum, so
\[ d\Omega = \left( \sum_{i=1}^{q} \alpha_{ii} \right) \wedge \Omega. \]

**Lemma 2.10.** For all $\alpha$ as above $(d\alpha)^{q+1} = 0$.

**Proof.**
\[ 0 = d^2 \Omega = d\alpha \wedge \omega - \alpha \wedge d\Omega = d\alpha \wedge \Omega + \alpha \wedge \alpha \wedge \Omega = d\alpha \wedge \Omega. \]
Write $d\alpha$ using basis of 2-forms extending $\{\omega_1, \ldots, \omega_q\}$
\[ d\alpha = \sum_{1 \leq i < j \leq n} f_{ij} \omega_i \wedge \omega_j. \]
Now take exterior product with $\Omega = \omega_1 \wedge \ldots \wedge \omega_q$
\[ \sum_{1 \leq i < j \leq n} f_{ij} \omega_i \wedge \omega_j \wedge \omega_1 \wedge \ldots \wedge \omega_q = 0. \]
If at least one of $i, j \in \{1, \ldots, q\}$ then corresponding summand is 0. Hence
\[ \sum_{q+1 \leq i < j \leq n} f_{ij} \omega_i \wedge \omega_j \wedge \omega_1 \wedge \ldots \wedge \omega_q = 0, \]
so
\[ f_{ij} = 0 \] for \( q + 1 \leq i < j \leq n. \)

Now we can write
\[ \sum_{i<j; \text{ at least one } \leq q} f_{ij} \omega_i \land \omega_j = \sum_{j=1}^{q} \alpha_j \land \omega_j \in S(E), \]
and
\[ (d\alpha)^{q+1} = \sum_{i_1j_1 \ldots i_qj_q} \omega_{i_1} \ldots \omega_{i_q} \land \omega_{j_1} \ldots \land \omega_{j_q} = 0. \]

We just proved that form \( \eta = \alpha \land (d\alpha)^q \) is closed.

**Lemma 2.11.** The class
\[ [\eta] \in H^{2q+1}(M, \mathbb{R}) \]
is independent on all choices involved in definition.

**Proof.** First assume that \( \Omega' = f\Omega \) for \( f > 0 \) everywhere. Then
\[ d\Omega' = f d\Omega + df \land \Omega = \alpha \land \Omega + df \land \Omega = \alpha \land \Omega' + \frac{df}{f} \land \Omega' = \]
\[ = (\alpha + d(\log f)) \land \Omega' = \alpha' \land \Omega'. \]
Hence
\[ \Omega' \land (d\Omega')^q = (\alpha + d(\log f)) \land (d\alpha)^q = \alpha \land (d\alpha)^q \land (d(\log f)(d\alpha)^q), \]
so \( \eta \) and \( \eta' = \alpha' \land (d\alpha') \) differ by boundary.

Now assume that \( d\Omega = \alpha' \land \Omega, \beta = \alpha - \alpha' \) such that \( \beta \land \Omega = 0. \) Hence \( \beta \in S(E), \) and recall that also \( d\alpha, d\alpha' \in S(E). \) Then we have
\[ \eta' = \alpha' \land (d\alpha')^q = (\alpha + \beta) \land ((d\alpha)^q + d\beta \land \sigma) \]
with
\[ \sigma = \sum_{i=0}^{q-1} c_i(d\alpha^i) \land (d\beta)^{q-i-1} \in S(E)^{q-1}, \quad \text{and } d\sigma = 0. \]
Then
\[ \alpha' \land (d\alpha')^q = \alpha \land (d\alpha)^q + \alpha \land d\beta \land \sigma + \beta \land (d\alpha)^q + \beta \land d\beta \land \sigma, \]
where the last two summands belong to \( S(E)^{q+1} = 0, \) so in fact we have
\[ \alpha' \land (d\alpha')^q = \alpha \land (d\alpha)^q + \alpha \land d\beta \land \sigma = \]
\[ = \alpha \land (d\alpha)^q + \alpha \land d(\beta \land \sigma) = \alpha \land (d\alpha)^q - d(\alpha \land \beta \land \sigma) + d\alpha \land \beta \sigma, \]
where the last summand is from \( S(E)^{q+1} = 0. \) Again we see, that \( \eta' - \eta \) is a boundary.

**Definition 2.12.** The class
\[ gv(F) := [\eta] \in H^{2q+1}(M; \mathbb{R}) \]
is called Godbillon-Vey class of a manifold with foliation \( (M, F). \)
Remark 2.13. Nonorientable case. Lift $\mathcal{F}$ to $\tilde{\mathcal{F}}$ in $\tilde{M} = $ orientable double covering with $\gamma = \text{the generator of } \mathbb{Z}/2$. Replacing $\tilde{\Omega}$ by $\frac{1}{2}(\tilde{\Omega} - \gamma^* \tilde{\Omega}) \neq 0$ if needed, we can always assume 

$$\gamma^* (\tilde{\Omega}) = -\tilde{\Omega}.$$ 

Then 

$$d\tilde{\Omega} = \alpha \wedge \tilde{\Omega},\quad \text{and} \quad d(\gamma^* \tilde{\Omega}) = \gamma^* (\tilde{\alpha}) \wedge \gamma^* (\tilde{\Omega}).$$ 

Hence 

$$d\tilde{\Omega} = \gamma^* (\tilde{\alpha}) \wedge \tilde{\Omega},\quad \text{and} \quad \frac{1}{2}(\tilde{\alpha} + \gamma^* (\tilde{\alpha}))$$ 

drops down to $M.$

2.4 Nontriviality of Godbillon-Vey class

On $G = \text{SL}(2, \mathbb{R})$, with $TG \simeq G \times \mathfrak{g}$, $(\mathfrak{g}$ - Lie algebra of $G = \text{traceless matrices})$ take the foliation given by the subbundle $E$ generated by the left invariant vector fields corresponding to 

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$ 

with 

$$[X,H] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2X.$$ 

The third basis element is 

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$ 

with 

$$[Y,H] = 2Y, \quad [X,Y] = H.$$ 

Take the dual basis $\{\zeta, \eta, \chi\}$ of $\mathfrak{g}^*$ and extend them as left-invariant 1-forms. Then $\eta$ defines $\mathcal{F}$ (i.e. $E = \ker \eta$). One has 

$$d\chi = a \chi \wedge \zeta + b \chi \wedge \eta + c \zeta \wedge \eta,$$

$$b = d\chi(H,Y) = -\chi([H,Y]) = 2\chi(Y) = 0$$

$$c = d\chi(X,Y) = -\chi([X,Y]) = -\chi(H) = -1$$

$$a = d\chi(H,X) = \chi([X,H]) = -2\chi(X) = 0,$$

hence 

$$d\chi = -\zeta \wedge \eta.$$ 

Similarly 

$$d\zeta = -2\chi \wedge \zeta,$$

$$d\eta = 2\chi \wedge \eta.$$ 

The last implies 

$$\alpha = 4\chi \wedge d\chi = -4\chi \wedge \zeta \wedge \eta.$$ 

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The form $\alpha$ drops down to $M = \Gamma \backslash G$ for any $\Gamma$ cocompact giving a volume form, hence

$$[\alpha_\Gamma] = \text{generator of } H^3(M; \mathbb{R}).$$

More precisely, let $\Sigma_g$ be the Riemann surface of genus $g \geq 2$. Then its universal cover is the upper half plane

$$\mathbb{H} = \text{SL}(2, \mathbb{R})/\text{SO}(2),$$
on which $\Gamma = \pi_1(\Sigma_g)$ acts by Mobius transformation

$$\Gamma \subset \text{PSL}(2, \mathbb{R}), \quad z \mapsto \frac{az + b}{cz + d}.$$

Let $\tilde{\Gamma}$ be the double cover of $\Gamma$. Then $\tilde{\Gamma}$ is cocompact. Morover $M \simeq S^1\Sigma_g$ (unit tangent bundle), hence

$$[\alpha_{\Gamma}]([M]) = 4 \int_{S^1\Sigma_g} \zeta \wedge \eta \wedge \chi = 4\pi \int_{\Sigma_g} \zeta \wedge \eta = 4\pi \text{Area}(\Sigma_g) =$$

$$= -4\pi \int_{\Sigma_g} K d\sigma = -8\pi^2(2 - 2g).$$

### 2.5 Naturality under transversality

Let $\phi: N \to M$, $E \subset TM$ integrable subbundle, $\mathcal{F}$- codimension $q$ foliation, $\tau\mathcal{F} = E$.

If $V \to M$ is a vector bundle, then for each invariant polynomial $P \in \mathcal{I}(\mathfrak{gl}_q(\mathbb{R}))$ of degree $k$, we have a class $P(V) \in H^{2k}(M; \mathbb{R})$. It behaves naturally with respect to pullback

$\phi^*(V) \to V$

$\phi \downarrow \phi$

$\pi$

$P(\phi^*(V)) = \phi^*(P(V)).$

By Bott’s vanishing theorem (2.7), all classes for $Q = TM/E$ are 0 if $k > q$. The Godbillon-Vey class $\text{gv}(M, \mathcal{F}) \in H^{2q+1}(M; \mathbb{R})$ is a nontrivial invariant.

**Definition 2.14.** We say that $\phi$ is transversal to $E$ (or to $\mathcal{F}$), $\phi \pitchfork E$, if for each $x \in N$

$$T_{\phi(x)}M = \phi_*(T_xN) \oplus E_{\phi(x)}.$$

Equivalently

$$\pi \circ \phi_*: T_xN \to T_{\phi(x)}M/E$$

is surjective.

**Lemma 2.15.** $\tilde{E} := \phi^{-1}_*(E)$ is involutive, hence defining a foliation $\tilde{\mathcal{F}} = \phi^{-1}(\mathcal{F})$, whose leaves are the connected components of $\phi^{-1}(L)$, $L \subset \mathcal{F}$.

**Proof.** (Short) Let $E = \tau\mathcal{F}$ be given by a cocycle $\{(U_i, f_i, g_{ij}) \mid i, j \in I\}$, $f_i: U_i \to \mathbb{R}^q$ submersions, $g_{ij}: f_j(U_i \cap U_j) \simeq f_i(U_i \cap U_j)$. Then $\{(\phi^{-1}(U_i), f_i \circ \phi, g_{ij}) \mid i, j \in I\}$ define $\tilde{\mathcal{F}}$. □
Proof. (More useful) Any map $\phi$ can be decomposed as a composition
\[
N \xrightarrow{id \times \phi} N \times M \xrightarrow{pr_M} M,
\]
\[x \mapsto (x, \phi(x)); \quad (x, y) \mapsto y.
\]
It is sufficient to prove the lemma for
(a) $id \times \phi$ - injective immersion,
(b) $pr_M$ - projection.

For each map in this composition the statement is obvious.
(a) $\tilde{E} = E \cap TN$,
(b) $\tilde{E} = TN \oplus E$.

Definition 2.16. A characteristic class for foliation $\mathcal{F}$ is an assignment
\[(M, \mathcal{F}) \mapsto \gamma(M, \mathcal{F}) \in H^*(M; \mathbb{R})\]
such that if $\phi: N \to M$ is transversal to $\mathcal{F}$, then
\[
\gamma(N, \phi^*(\mathcal{F})) = \phi^*(\gamma(M, \mathcal{F})).
\]

Example 2.17. If $(M, \mathcal{F})$ is transversally oriented, i.e. there exists nowhere zero section $\Omega$ of $\Lambda^qQ$, then we have Godbillon-Vey class. On local chart $U$
\[
\Omega = \omega_1 \wedge \ldots \wedge \omega_q, \quad \{\omega_1, \ldots, \omega_q\} \quad \text{generators of } S(E|_U),
\]
\[d\Omega = \alpha \wedge \Omega, \quad gv(M, \mathcal{F}) = [\alpha \wedge (d\alpha)^q] \in H^{2q+1}(M; \mathbb{R}).\]

For $\phi: N \to M$
\[
\{\phi^*(\omega_1), \ldots, \phi^*(\omega_q)\} \quad \text{generators of } S(\phi^*(E)|_{\phi^{-1}(U)})
\]
and therefore
\[d\phi^*(\Omega) = \phi^*(d\Omega) = \phi^*(\alpha) \wedge \phi^*(\Omega),
\]
and thus
\[gv(N, \phi^*(\mathcal{F})) = \phi^*(\alpha) \wedge (d\phi^*(\alpha))^q = \phi^*(\alpha \wedge (d\alpha)^q) = \phi^*(gv(M, \mathcal{F})).\]

Example 2.18. Pontryagin classes are characteristic classes of for foliation, since for $P \in T^k(gl_q(\mathbb{R}))$ we have
\[P(\phi^*(\mathcal{F})) = \phi^*(P(\mathcal{F})),\]
where $P(\mathcal{F}) = P(Q)$ for $Q = TM/\tau\mathcal{F}$.
2.6 Transgressed classes

Let \((M, \mathcal{F})\) be a manifold with foliation, \(\nabla_0, \nabla_1\) two connections on \(Q = TM/E, E = \tau \mathcal{F}\). Then 
\[
\nabla_1 - \nabla_0 = \alpha \in \Omega^1(M, \text{End}(E)).
\]
Let \(\nabla_t := t\nabla_1 + (1-t)\nabla_0\) be linear homotopy between connections, and \(R_0, R_1, R_t\) corresponding curvatures. Then by the theorem of Chern-Weil (2.2) for \(P \in T^k(\mathfrak{gl}_q(\mathbb{R}))\)
\[
P(R_1) - P(R_0) = dTP(\nabla_1, \nabla_0), \text{ where}
\]
\[
T P(\nabla_1, \nabla_0) := k \int_0^1 P(\alpha, R_t, \dots, R_t) dt.
\]
Let \(\nabla_1 = \nabla^\flat\) be the \(E\)-flat connection (or Bott connection) (def. (2.3)), i.e.
\[
\nabla^\flat_X(\pi(Y)) = \pi([X,Y]), \quad \forall X \in \mathcal{S}(E), \pi: TM \to TM/E = Q.
\]

The corresponding curvature satisfies (lemma (2.6))
\[
\tilde{R}^\flat(X_1, X_2) = 0, \quad \forall X_1, X_2 \in \mathcal{S}(E).
\]

As a second connection \(\nabla_0\) we take metric (or Riemannian) connection \(\nabla^\sharp\), i.e.
\[
X(s_1, s_2) = (\nabla^\sharp_X s_1, s_2) + (s_1, \nabla^\sharp_X s_2),
\]
for \(s_1, s_2 \in \mathcal{S}(Q)\). Then

- \(P(\tilde{R}^\flat) = 0\) if \(k > q\), by Bott’s theorem (2.7),
- \(P(\tilde{R}^\sharp) = 0\) if \(k\) is odd, by lemma (2.3).

In particular for \(k > q\) odd form \(TP(\nabla^\flat, \nabla^\sharp)\) is closed, \(dTP(\nabla^\flat, \nabla^\sharp) = 0\), so
\[
TP(M, \mathcal{F}) := [TP(\nabla^\flat, \nabla^\sharp)] \in H^{2k-1}(M; \mathbb{R}).
\]

**Definition 2.19.** We call \(TP(M, \mathcal{F})\) a transgressed class.

**Proposition 2.20.** For foliation \(\mathcal{F}\) on a manifold \(M\) and \(P \in T^k(\mathfrak{gl}_q(\mathbb{R}))\), \(k > q = \text{dim } TM/\tau \mathcal{F}\), class \([TP(M, \mathcal{F})] \in H^{2k-1}(M; \mathbb{R})\) is independent of choices \(\nabla^\flat\) and \(\nabla^\sharp\), and therefore is an invariant of foliation.

**Proof.** Let \(\nabla^\flat, \nabla^\sharp\), \(i = 0, 1\) be two different choices of connections, and let
\[
i^n \nabla^\flat := \psi(t)^1 \nabla^\flat + (1 - \psi(t))0 \nabla^\flat,
i^n \nabla^\sharp := \psi(t)^1 \nabla^\sharp + (1 - \psi(t))0 \nabla^\sharp,
\]
where in both cases \(\psi: [0,1] \to [0,1]\) is a smooth function such that \(\psi \equiv 0\) near 0 and \(\psi \equiv 1\) near 1.

Now take the bundle \(\tilde{E} = E \oplus \mathbb{R}\) on \(M \times \mathbb{R}\) (as a integrable bundle of foliation on \(M \times \mathbb{R}\)). On the quotient \(pr^*_M(Q)\) we define the connections \(\tilde{\nabla}^\flat\) and \(\tilde{\nabla}^\sharp\).
\[
pr^*_M(Q) = T(M \oplus \mathbb{R})/\tilde{E} \quad Q = TM/\tau \mathcal{F}
\]
\[
\begin{array}{ccc}
M \times \mathbb{R} & \xrightarrow{pr_M} & M \\
\downarrow & & \downarrow \\
pr^*_M(Q) & & \tilde{E}
\end{array}
\]

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Sections of bundles over $M \times \mathbb{R}$ can be represented as follows

$$S(T(M \times \mathbb{R})) = \{ f(x,s)Y + g(x,s)\frac{\partial}{\partial s} \mid Y \in S(TM), f, g \in C^\infty(M \times \mathbb{R}) \}.$$ 

$$S(pr^*_M(Q)) = \{ f(x,s)\pi(Y) \mid Y \in S(TM), \pi: TM \to Q, f \in C^\infty(M \times \mathbb{R}) \}$$

It suffices to define

$$\nabla_{(x, \frac{\partial}{\partial t})} (\pi(Y)) := \nabla_X (\pi(Y)),$$

for $\nabla = \nabla^\flat$ or $\nabla^\sharp$.

We have

$$\nabla_X (f(x,s)\pi(Y)) = X(f) \pi(Y) + f^* \nabla_X (\pi(Y)),$$

$$\nabla_{\frac{\partial}{\partial t}} (f(x,s)\pi(Y)) = \frac{\partial f}{\partial s} \pi(Y),$$

where $f^* \nabla^\flat = s^0 \nabla^\flat + (1-s)^0 \nabla^\flat$, $f^* \nabla^\sharp = s^0 \nabla^\sharp + (1-s)^0 \nabla^\sharp$. Using inclusions $i_s: M \to M \times \mathbb{R}$, $i_s(x) = (x, s)$, we can write

$$i_0^*(\nabla^\flat) = 0 \ R^\flat, \ i_1^*(\nabla^\flat) = 1 \ R^\flat$$

and analogously for $\nabla^\sharp, R^\sharp$. Similarly

$$i_0^*(\nabla) = 0 \ \alpha, \ i_1^*(\nabla) = 1 \ \alpha$$

for corresponding differences $0 \alpha = 0 \ \nabla^\flat - 0 \ \nabla^\sharp$ and $1 \alpha = 1 \ \nabla^\flat - 1 \ \nabla^\sharp$. Hence

$$i_0^*(TP(\nabla^\flat, \nabla^\sharp)) = TP(0 \ \nabla^\flat, 0 \ \nabla^\sharp),$$

and

$$i_1^*(TP(\nabla^\flat, \nabla^\sharp)) = TP(1 \ \nabla^\flat, 1 \ \nabla^\sharp).$$

Note that $\nabla^\flat$ is $\tilde{E}$-flat, and $\nabla^\sharp$ is Riemannian for $pr^*_M(Q)$.

The proof is completed by the elementary lemma (homotopy invariance of de Rham cohomology)

**Lemma 2.21.** Let $\omega \in \Omega^k(M \times \mathbb{R}), d\omega = 0$. Then $i_1^*(\omega) - i_0^*(\omega)$ is exact.

**Proof.** We can write

$$\omega = \pi^*(\alpha) \wedge f(x,t) dt + g(x,t) \pi^*(\beta),$$

with $\alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M)$.

One has

$$L_{\partial_t}(\omega) = dt \partial_t + \iota_{\partial_t} d\omega = L_{\partial_t}(\omega) = d((-1)^{k-1} f(x, t) pr^*_M(\alpha)) =$$

$$= (-1)^{k-1} f(x, t) d pr^*_M(\alpha) + pr^*_M(\alpha) \wedge dx + d_x f + pr^*_M(\alpha) \wedge \partial_t f dt,$$

where $\partial_t := \frac{\partial}{\partial t}$. On the other hand

$$L_{\partial_t} |_{s=t_0} (\omega) = \frac{\partial}{\partial s} |_{s=t_0} (i_s(pr^*_M(\alpha) \wedge f(x,t) dt + g(x,t) pr^*_M(\beta))) =$$

$$= \partial_t f(x,t) |_{t_0} pr^*_M(\alpha) \wedge dt + \partial_t g(x,t) |_{t_0} pr^*_M(\beta).$$

Comparing both sides one gets

$$\partial_t g(x,t) \wedge pr^*_M(\beta) = (-1)^{k-1} (f(x,t) d pr^*_M(\alpha) + d_x f(x,t) \wedge pr^*_M(\alpha)) =$$
\[ = (-1)^{k-1} d_x (f(x,t) \text{pr}_M^*(\alpha)). \]

Hence
\[ g(x,1) \text{pr}_M^*(\beta) - g(x,0) \text{pr}_M^*(\beta) = (-1)^{k-1} d_x \left( \int_0^1 f(x,t) dt \cdot \text{pr}_M^*(\alpha) \right), \]
so
\[ i_1^*(\omega) - i_0^*(\omega) = d \left( (-1)^{k-1} \int_0^1 f(x,t) dt \cdot \alpha \right). \]

\[ \square \]

\textbf{Proposition 2.22.} For any \( P \in \mathcal{I}^k(\mathfrak{gl}_n(\mathbb{R})) \) with \( k > q \) odd, \( TP(M,F) \) is a characteristic class.

\textit{Proof.} It is sufficient to prove the naturality in two special cases

1. \( i: N \to M \) is injective immersion,
2. \( p: N \times M \to M \) a projection.

Case. 1 We have \( i^*(E) = E \cap TN, i^*(Q) = Q|_N \), hence \( \nabla^\sharp, \nabla^\flat \) restrict to the same kind of connections. Thus one has

\[ TP(N, i^*(F)) = i^*(TP(M,F)). \]

Case. 2 We lift \( \nabla^\flat, \nabla^\sharp \) to the same kind of connections on \( N \times M \). \( \tilde{R}_t = p^*(R_t), \tilde{\alpha} = p^*(\alpha). \)

\[ \square \]

\textbf{Definition 2.23.} Two vector bundles \( E_0, E_1 \subset TM \) of codim = \( q \) are transversally homotopic if there exists \( \tilde{E} \subset T(M \times \mathbb{R}) \) of codim = \( q \), such that

1. \( \tilde{E} \) is involutive,
2. \( \tilde{E} \) is transversal to \( M \times \{0\} \) and \( M \times \{1\} \),
3. \( i_0^*(\tilde{E}) = E_0 \) and \( i_1^*(\tilde{E}) = E_1. \)

\textbf{Proposition 2.24.} The class \( TP(M,F) \) depends only on transverse homotopy class of foliation \( F \).
Chapter 3
Weil algebras

3.1 The truncated Weil algebras and characteristic homomorphism

The set of invariant polynomials \( I(\mathfrak{gl}_q(\mathbb{R})) \) is generated by \( P_k(A) := \text{tr}(A^k), \ A \in \mathfrak{gl}_q(\mathbb{R}). \) Alternatively we have

\[
\det(I + tA) = \sum_{i=0}^{q} c_i(A)t^i.
\]

Coefficients \( c_i(A) \) are symmetric functions of eigenvalues. If

\[
A \sim \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_q \end{pmatrix}
\]

then

\[
\det(I + tA) = (1 + t\lambda_1)(1 + t\lambda_2)\ldots(1 + t\lambda_q) = 1 + t(\lambda_1 + \lambda_2 + \ldots + \lambda_q) + t^2(\sum \lambda_i\lambda_j) + \ldots + t^q\lambda_1\lambda_2\ldots\lambda_q.
\]

\[c(A) := \det(I + A) = 1 + c_1(A) + \ldots + c_q(A),\]

\[c(A \oplus B) = c(A)c(B).\]

The set \( I(\mathfrak{gl}_q(\mathbb{R})) \) can be presented as polynomial ring

\[I(\mathfrak{gl}_q(\mathbb{R})) = \mathbb{R}[c_1, \ldots, c_q].\]

For manifold with foliation \((M, \mathcal{F}), Q = TM/E, E = \tau\mathcal{F},\) we have

\[c_k(R^p) = 0, \ \forall k > q.\]

Moreover for each \( P \in \mathbb{R}^k[c_1, \ldots, c_q], \ k > q\)

\[P(R^p) = 0 \in \Omega^{2k}(M).\]

Define

\[\mathbb{R}[c_1, \ldots, c_q]_q := \mathbb{R}[c_1, \ldots, c_q]/(\text{weight} > 2q), \ \deg(c_i) = 2i.\]
For any connection $\nabla$ on $E$ we have a map

$$\lambda_E(\nabla) : \mathbb{R}[c_1, \ldots, c_q] \to \Omega^\bullet(M),$$

$$\lambda_E(\nabla)(P) := P(\nabla^2).$$

**Proposition 3.1.** 1. $\lambda_E(\nabla^b)$ annihilates all polynomials of degree $> q$, so it induces a map

$$\lambda_E(\nabla^b) : \mathbb{R}[c_1, \ldots, c_q] \to \Omega^\bullet(M).$$

2. $\lambda_E(\nabla^t)$ annihilates all polynomials of odd degree, in particular

$$\lambda_E(\nabla^t)(c_{2i-1}) = 0.$$

3. There is a third map

$$T\lambda_E(\nabla^b, \nabla^t) : \mathbb{R}[c_1, \ldots, c_q] \to \Omega^\bullet(M)$$

satisfying

$$dT\lambda_E(\nabla^b, \nabla^t)(P) = \lambda_E(\nabla^b)(P) - \lambda_E(\nabla^t)(P).$$

In particular

$$dT\lambda_E(\nabla^b, \nabla^t)(c_{2i-1}) = \lambda(\nabla^b)(c_{2i-1}).$$

This can be summarized in the following cochain complex. First form a differential graded algebra (DGA)

$$WO_q := \Lambda(u_1, u_3, \ldots, u_{2l-1}) \otimes \mathbb{R}[c_1, \ldots, c_q],$$

where the first algebra in the tensor product is an exterior algebra generated by elements $u_{2i-1}$ of degree $4i - 3$, and $l$ is maximal integer such that $2l - 1 \leq q$. Generators of second algebra $c_j$ have degree $2j$, and this is a quotient of polynomial algebra by the ideal of polynomials of degree $> q$ (weight $> 2q$). Now define $d: WO_q \to WO_q$ as the differential of degree 1 given on generators by the formula

$$du_{2i-1} = c_{2i-1}, \quad 1 \leq i \leq l,$$

$$dc_j = 0, \quad 1 \leq i \leq q.$$

**Definition 3.2.** Define a map $\lambda_E : WO_q \to \Omega^\bullet(M)$ by

$$\lambda_E(u_{2i-1}) := T\lambda_E(\nabla^b, \nabla^t)(c_{2i-1}),$$

$$\lambda_E(c_j) := \lambda_E(\nabla^b)(c_j), \quad 1 \leq j \leq q.$$

Then $\lambda_E : WO_q \to \Omega^\bullet(M)$ is a map of DGA’s, hence it induces a map

$$\lambda_E^* : H^*(WO_q) \to H^*(M; \mathbb{R})$$

of cohomology algebras.

We call $\lambda_E^*$ a characteristic map in analogy to

$$\chi_E : H^*(B\operatorname{GL}_n(\mathbb{R})) = \mathcal{I}(\mathfrak{gl}_n(\mathbb{R})) \to H^*(M; \mathbb{R})$$

for a $n$-dimensional vector bundle $E \to M$. 19
Theorem 3.3 (Bott). 1. \( \lambda^*_E \) depends only on \( E \), and not on the choice of connections.

2. \( \lambda^*_E \) is natural, i.e. for \( \phi: N \to M \), \( \phi \pitchfork \mathcal{F} \), one has
   \[
   \lambda^*_{\phi^*(E)} = \phi^* \circ \lambda^*_E.
   \]

3. \( \lambda^*_E \) depends only on the transverse homotopy class of \( E \) (def. (2.23)).

Proof. Theorem has essentially been proved.

1. This has been proved in proposition (2.20).

2. This has been proved in proposition (2.22).

3. The same proof as in proposition (2.20) and lemma (2.21) with \( \tilde{\nabla}_t \) on \( M \times I \) inducing \( \nabla^0_t \) on \( E_0 \) and \( \nabla^1_t \) on \( E_1 \).

Example 3.4 (WO₁ and Godbillon-Vey class). For \( q = 1 \) we have
   \[
   WO_1 = \Lambda \langle u_1 \rangle \otimes \mathbb{R}[c_1],
   \]
   hence \( \{1, u_1, c_1, u_1\} \) form a \( \mathbb{R} \)-basis and \( du_1 = c_1, \ dc_1 = 0 \). Clearly
   \[
   H^0(WO_1) = \mathbb{R} \cdot 1, \\
   H^1(WO_1) = 0, \\
   H^2(WO_1) = 0, \\
   H^3(WO_1) = \mathbb{R} \cdot u_1 c_1.
   \]
   Let \((M, E)\) be a manifold with codim = 1 foliation \( \mathcal{F}, \tau \mathcal{F} = E \), and assume that \( Q = TM/E \) is trivializable (i.e. \( E \) transversely oriented).
   \[
   \lambda_E(c_1) = \lambda_E(\nabla^b)(c_1), \\
   \lambda_E(u_1) = T\lambda_E(\nabla^b, \nabla^b)(c_1).
   \]
   Let \( \Omega \in \Omega^1(M) \) be the orientation form of \( Q^* \), so \( E = \ker \Omega \). Let \( Z \) be a vector field with \( \Omega(Z) = 1 \), which gives trivialization of \( Q \). Then
   \[
   TM = E \oplus \mathbb{R}Z.
   \]
   Let \( \Omega \) be defined by
   \[
   \Omega(X) = 0, \text{ for } X \in E, \\
   \Omega(Z) = 1.
   \]
   Then
   \[
   d\Omega = \alpha \wedge \Omega, \quad \alpha \in \Omega^1(M).
   \]
   Form \( \alpha \) defines a Bott connection by
   \[
   \nabla^b(\pi(Z)) = -\alpha \otimes \pi(Z), \\
   \nabla^*_X(\pi(Z)) = -\alpha(X)(\pi(Z)) = \pi([X, Z]).
   \]
Indeed, one has for all \( X \in E \)

\[
d\Omega(X, Z) = -\Omega([X, Z]) = -\Omega(\pi([X, Z])), \quad \text{and}
\]

\[
\alpha \wedge \Omega(X, Z) = \alpha(X)\Omega(Z) - \alpha(Z)\Omega(X) = \alpha(X).
\]

Thus

\[
\alpha(X) = -\Omega(\pi([X, Z])).
\]

Godbillon-Vey class is a class of \( \alpha \wedge d\alpha \) in \( H^3(M; \mathbb{R}) \). One the other hand one has

\[
(\nabla^h)^2(\pi(Z)) = \nabla^h(-\alpha \otimes \pi(Z)) = -d\alpha \otimes \pi(Z) + \alpha \wedge \alpha \otimes \pi(Z) =
\]

\[
d\alpha \otimes \pi(Z),
\]

hence

\[
R^h = d\alpha, \quad \text{so}
\]

\[
\lambda_E(c_1) = d\alpha.
\]

Define a Riemannian connection on \( Q \) by

\[
\nabla^\sharp_X(\pi(Z)) = 0, \quad \forall X \in E,
\]

\[
\nabla^\flat_Z(\pi(Z)) = 0, \quad \text{where } ||Z|| = 1.
\]

Then \( \nabla^h - \nabla^\sharp = -\alpha \in \Omega^1(M, \text{End}(Q)) = \Omega^1(M) \), hence

\[
\lambda_E(u_1) = T\lambda_E(\nabla^h, \nabla^\sharp)(c_1) = -\alpha.
\]

This implies

\[
\lambda_E(u_1c_1) = \alpha \wedge d\alpha = \text{gv}(M, \mathcal{F}).
\]

**Proposition 3.5.** If \( E = \tau F \) is of codim = \( q \), transversally oriented, then

\[
\lambda_E(u_1c_1^q) = \text{gv}(E).
\]

**Proof.** We have nonvanishing form \( \Omega \in S((Q^*)^q) \). Locally it can be written as

\[
\Omega = \omega_1 \wedge \ldots \wedge \omega_q,
\]

with \( \{\omega_1, \ldots, \omega_q\} \) generators of \( S(E) \). Write

\[
d\omega_i = \sum_j \alpha_{ij} \wedge \omega_j,
\]

and define \( \nabla^h : S(Q) \to S(T^*M \otimes Q) \) by

\[
\nabla^h(\pi(Z_i)) = -\sum_j \alpha_{ji} \otimes \pi(Z_j),
\]

where \( \{Z_1, \ldots, Z_q\} \) is a dual basis to \( \{\omega_1, \ldots, \omega_q\} \) on a complement of \( E \). One has for all \( X \in E \)

\[
d\omega_i(X, Z_k) = \sum_j (\alpha_{ij}(X)\omega_j(Z_k) - \alpha_{ij}(Z_k)\omega_j(X)).
\]
But
\[ d\omega_i(X, Z_k) = -\omega_i([X, Z_k]) = \pi([X, Z_k]), \]
and on the right hand side we have only \(\alpha_{ik}(X)\), so
\[ \pi([X, Z_k]) = \sum_i \alpha_{ik}(X)\pi(Z_i), \]
while
\[ \nabla_X^\flat(\pi(Z_k)) = -\sum_j \alpha_{jk}(X)\pi(Z_j) = \pi([X, Z_k]), \]
hence it is a Bott connection. Its curvature is
\[
(\nabla^\flat)^2(\pi(Z_i)) = -\sum_j \nabla^\flat(\alpha_{ij} \otimes \pi(Z_j)) = \\
= -\sum_j d\alpha_{ji} \otimes \pi(Z_j) + \sum_j \alpha_{ji}(-\sum_k \alpha_{kj} \otimes \pi(Z_k)) = \\
= -\sum_k (d\alpha_{ki} - \sum_j \alpha_{kj} \wedge \alpha_{ji})\pi(Z_k),
\]
i.e.
\[ R = d\alpha - \alpha \wedge \alpha. \]
This implies
\[ c_1(R) = \text{tr}(d\alpha) - \text{tr}(\alpha \wedge \alpha) = \text{tr}(d\alpha) = d(\text{tr}\alpha), \]
hence
\[ c_1(R)^q = d(\text{tr}\alpha)^q. \]

Take Riemannian connection given by an orthogonal matrix form
\[ \nabla^\sharp(\pi(Z_i)) = \sum_j \beta_{ij} \otimes \pi(Z_j). \]
Now
\[ (\nabla^\flat - \nabla^\sharp)(\pi(Z_i)) = \sum_j (\alpha_{ij} + \beta_{ij}) \otimes \pi(Z_j), \]
hence
\[ \nabla^\flat - \nabla^\sharp = -\alpha - \beta, \quad \text{tr}\beta = 0 \]
so the transgressed form is
\[ Tc_1(\alpha + \beta) = \text{tr}\alpha. \]
Now
\[ \text{gv}(E) = [\text{tr}\alpha \wedge (\text{tr}(d\alpha))^q] = [u_1c_1(R)^q]. \]
3.2 $W_q$ and framed foliations

**Definition 3.6.** Differential graded algebra $W_q$

$$W_q := \Lambda\langle u_1, \ldots, u_q \rangle \otimes \mathbb{R}[c_1, \ldots, c_q]_q$$

$$du_i = c_i, \quad dc_i = 0, \quad \forall i = 1, \ldots, q.$$ 

These algebras are useful for foliation $(M, \mathcal{F})$ with $Q$ trivializable, when one can transgress to a flat Riemannian connection and get

$$\mu_E : W_q \rightarrow \Omega^\bullet(M),$$

$$\mu_E(u_i) := T\lambda_E(\nabla^p, \nabla^L^0)(c_i),$$

$$\mu_E(c_i) := \lambda_E(\nabla^p)(c_i).$$

Notation: for $i_1 < \ldots < i_r$, $j_1 \leq \ldots \leq j_s$ we denote

$$u_{ICJ} = u_{i_1} \ldots u_{i_r}, c_{j_1} \ldots c_{j_s}.$$

**Proposition 3.7.** The elements

(a)$$1 \cup \{ u_{ICJ} \mid |J| \leq q, i_1 + |J| > q, i_1 \leq j_1 \}$$

form a basis of $H^\ast(W_q)$.

(b)$$1 \cup \{ u_{ICJ} \mid i_k \text{ odd, } |J| < q, i_1 + |J| > q, \text{ and } \left\{ \begin{array}{l} \text{if } r = 0 \text{ then all } j_k \text{ even,} \\
\text{if } r \neq 0 \text{ then } i_1 \leq \min_{\text{odd}}\{j_k\} \end{array} \right\}$$

form a basis of $H^\ast(WQ_q)$.

**Proof.** (sketch)

Ad.(a)

$$d(u_{ICJ}) = \sum_{k=1}^{r} (-1)^{k-1} u_{i_1} \ldots du_{i_k} \ldots u_{i_r} c_{J} =$$

$$= \sum_{k=1}^{r} (-1)^{k-1} u_{i_1} \ldots \widehat{u_{i_k}} \ldots u_{i_r} c_{i_k} c_{J} = 0,$$

because $\deg c_{i_k} c_J \geq 2(|J| + i_1) > 2q$.

Ad.(b) If $r = 0$ then $d(c_J) = 0$. The case $r \neq 0$ is treated as above.

Consequences of (a) for $H^\ast(W_q)$.

1. $$\deg(u_{ICJ}) = (2i_1 - 1) + \ldots + (2i_r - 1) + (2j_1 + \ldots + 2j_s) \leq$$

$$\leq 2(1 + \ldots + q) - q + 2|J| \leq q(q + 1) - q + 2q = q^2 + 2q.$$

Hence

$$H^m(W_q) = 0, \quad \text{for } m > q^2 + 2q.$$
2. On the other hand
\[ \deg(u_Ic_J) \geq 2|J| > 2q, \]

hence
\[ H^m(W_q) = 0, \quad \text{for } 1 \leq m < 2q. \]

With a little more work we can eliminate \( m = 2q \) which can occur only if \(|I|\) even.

3. The product structure is trivial.

4. In \( H^{2q+1}(W_q) \) the classes \( u_1c_1^{\alpha_1} \ldots c_k^{\alpha_k} \) with \( \sum_{i=1}^k \alpha_i = q \) are linearly independent.

Similar conclusions hold for \( H^*(WO_q) \):

1. 
\[ H^m(WO_q) = 0, \quad \text{for } m > q^2 + 2q. \]

2. For \( m \leq 2q \) one gets the Pontryagin classes
\[ \{1, p_1, \ldots, p_{\left\lfloor \frac{q}{2} \right\rfloor}\}. \]

3. The product structure is trivial in 'high degree'.

4. In \( H^{2q+1}(WO_q) \) the classes \( u_1c_1^{\alpha_1} \ldots c_k^{\alpha_k} \) with \( \sum_{i=1}^k \alpha_i = q \) are linearly independent.
Chapter 4

Gelfand-Fuks cohomology

4.1 Cohomology of Lie algebras

Recall the formula for the exterior derivation

\[ d: \Omega^p(M) \to \Omega^{p+1}(M) \]

\[ d\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \omega(X_0, \ldots, \widehat{X}_i, \ldots, X_p) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_p). \]

\[ H^\ast(\Omega^\ast(M), d) = H^\ast_{dR}(M; \mathbb{R}). \]

We can view \( \Omega^\ast(M) \) as a \( C^\infty(M) \) linear homomorphisms

\[ \Omega^\ast(M) \simeq \text{Hom}_{C^\infty(M)}(\Lambda^\ast V_M, C^\infty(M)), \]

where \( V_M \) is a Lie algebra of vector fields on \( M \) with

\[ [X, Y] = XY - YX. \]

More general context consists of

- \( \mathfrak{g} \) - a Lie algebra of finite dimension over a field \( k \),
- \( A \) - \( \mathfrak{g} \)-module
- Cochains \( C^\ast(\mathfrak{g}; A) := \text{Hom}_k(\Lambda^\ast \mathfrak{g}, A) \) with differential

\[ d: C^p(\mathfrak{g}; A) \to C^{p+1}(\mathfrak{g}; A), \]

given by the same formula as above.

- Cohomology

\[ H^\ast(\mathfrak{g}; A) := H^\ast(C^\ast(\mathfrak{g}; A), d). \]
Relative Lie algebra cohomology is defined as follows. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra. Define relative cochains as

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) := \{ c \in C^\bullet(\mathfrak{g}; A) \mid \iota_X c = 0 \text{ and } \iota_X dc = 0 \forall X \in \mathfrak{h} \}.$$ 

By definition it is a subcomplex and its cohomology is

$$H^\bullet(\mathfrak{g}, \mathfrak{h}; A) := H^\bullet(C^\bullet(\mathfrak{g}, \mathfrak{h}; A), d).$$

Since

$$L_X = d\iota_X + \iota_X d, \quad L_X \omega = d\iota_X \omega + \iota_X d\omega = 0,$$

alternatively we can put

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) := \{ c \in C^\bullet(\mathfrak{g}; A) \mid c \text{ basic i.e. } \iota_X c = 0 \text{ and } L_X c = 0 \forall X \in \mathfrak{h} \}.$$ 

One has

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) = \text{Hom}_k(\Lambda^\bullet(\mathfrak{g}/\mathfrak{h}), A)^\mathfrak{h}.$$ 

Slightly more generally, if $H$ is a Lie group with $\mathfrak{h} = \text{Lie}(H)$, acting on $\mathfrak{g}$ and $A$ such that, the differential of the action on $\mathfrak{g}$ is $\text{ad}_{\mathfrak{g}}$, then

$$C^\bullet(\mathfrak{g}, H; A) := \{ c \in \text{Hom}_H(\Lambda^\bullet \mathfrak{g}, A) \mid \iota_X c = 0 \forall X \in \mathfrak{h} \},$$

and its cohomology is

$$H^\bullet(\mathfrak{g}, H; A).$$

**Example 4.1.** Let $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{R})$. Its complexification is $\mathfrak{g}_\mathbb{C} := \mathfrak{sl}_n(\mathbb{C})$. We have

$$H^\bullet(\mathfrak{g}_\mathbb{C}) = H^\bullet(\mathfrak{g}) \otimes \mathbb{C}.$$ 

Also one has for $\mathfrak{u}_n := \text{Lie}(U(n))$

$$H^\bullet(\mathfrak{u}_n(\mathbb{R})) = H^\bullet(\mathfrak{u}_n) = \Lambda(u_1, u_3, \ldots, u_{2l+1}), l = \left\lfloor \frac{n}{2} \right\rfloor.$$ 

Furthermore for $g \in U(n)$ and $k$ odd

$$d \text{tr}((g^{-1}dg)^k) = - \text{tr}((g^{-1}dg)^{k+1}) = 0.$$ 

The class $u_k := [\text{tr}((g^{-1}dg)^k)]$ is called a Chern-Simons class.

### 4.2 Gelfand-Fuks cohomology

Let $V_M$ be the algebra of vector fields on a manifold $M$, that is $\mathcal{S}(TM)$. $C^\infty$ topology on $V_M$ is given by $C^\infty$ convergence on compacta of the local components (which are functions), and their derivatives.

$$X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}, \quad f^i \in C^\infty(M).$$

**Definition 4.2.** Define the Gelfand-Fuks cohomology as the cohomology of the algebra $V_M$ continuous with respect to the $C^\infty$ topology on $V_M$

$$H^*_\text{GF}(V_M) := H^*_\text{cont}(V_M; \mathbb{R}).$$
Here \( C^\bullet_{\text{cont}}(V_M; \mathbb{R}) \) are continuous functionals on \( V_M \) with respect to \( C^\infty \) topology.

The remarkable fact [Gelfand-Fuks] is that \( H^*_G \) is finite dimensional. An important step in the proof of this is played by an algebra of formal vector fields on \( M \)

\[
\mathfrak{A}_n := \{ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i \in \mathbb{R}[[x^1, \ldots, x^n]] \}. 
\]

The dual algebra of vector fields

\[
V^*_M := \text{Hom}_{\text{cont}}(V_M, \mathbb{R})
\]

consists of distributions with compact support. The notion of support makes sense for the cochains

\[
C^\bullet_{\text{cont}}(V_M; \mathbb{R}) := \Lambda^\bullet V^*_M
\]

and is preserved by

\[
d: \Lambda^\bullet V^*_M \to \Lambda^{\bullet+1} V^*_M.
\]

In particular one can take for \( p_0 \in M \) the subcomplex

\[
\Lambda^\bullet V^*_M,p_0 := \text{distributions supported at } p_0.
\]

Then \( V^*_M,p_0 \) is a real vector space spanned by \( \nabla_{p_0} \) and its partial derivatives

\[
X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} 
\]

\[
X \mapsto (-1)^{|\alpha|} \frac{\partial^{|\alpha|} f^i}{\partial x^\alpha}.
\]

They only depend on the jet of \( X \) at \( p_0 \). Thus we are dealing with the continuous Lie algebra complex of

\[
\mathfrak{A}_n := \{ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i \in \mathbb{R}[[x^1, \ldots, x^n]] \}. 
\]

with the \( \mathcal{I} \)-adic topology (since the elements of the dual depend on finite set).

In \( \mathfrak{A}_n \) we have following forms

\[
\theta^i(X) := f^i(0), \ 1 \leq i \leq n,
\]

\[
\theta^j_j(X) := \frac{\partial f^i}{\partial x^j} \bigg|_{x=0}, \ 1 \leq i, j \leq n,
\]

\[
\theta^j_{jk}(X) := \frac{\partial^2 f^i}{\partial x^j \partial x^k} \bigg|_{x=0}, \ 1 \leq i, j, k \leq n,
\]

and generally for multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \)

\[
\theta^i_\alpha := (-1)^{|\alpha|} \frac{\partial^{|\alpha|} f^i}{\partial x^\alpha} \bigg|_{x=0}.
\]

We make \( \Lambda^\bullet \mathfrak{A}_n \) into a complex by defining the differential

\[
d\omega(X_0, \ldots, X_n) := \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n).
\]
1. The elements
\[ \{ \theta_i^\alpha \mid 1 \leq i \leq n, \alpha \in (\mathbb{Z}_+)^n \} \]
span \( C^1(\mathfrak{A}_n) = \mathfrak{A}_n^* \), hence generate all of
\[ C^\bullet(\mathfrak{A}_n) = \bigoplus_{k=0}^\infty \Lambda^k \mathfrak{A}_n^*. \]
Note that \( \theta_i^\alpha = \theta_i^\beta \) if \( \alpha = \beta \) as an unordered sets.

2. The Lie derivative
\[ \mathcal{L} \left( \frac{\partial}{\partial x^j} \right) \theta^i = \theta^j, \text{ and} \]
\[ \mathcal{L} \left( \frac{\partial}{\partial x^j} \right) \mathcal{L} \left( \frac{\partial}{\partial x^k} \right) \theta^i = \theta^i_{jk}, \text{ etc.} \]
Indeed
\[ \mathcal{L} \left( \frac{\partial}{\partial x^j} \right) \theta^i(X) = \left( \frac{d}{dt} \bigg|_{t=0} \tau^i \right)(X) = \theta^i \left( \frac{d}{dt} \bigg|_{t=0} \tau^j \right)(X) = \frac{d}{dt} \bigg|_{t=0} f^i(x^j - t, \ldots, x^n) = -\frac{\partial f^i}{\partial x^j} \bigg|_{x=0} = \theta^j (X). \]
In general
\[ \mathcal{L} \left( \frac{\partial}{\partial x^j} \right) \theta^i = \theta^i_{\alpha;j} \]
Since
\[ \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0, \]
we have
\[ \left[ \mathcal{L} \left( \frac{\partial}{\partial x^i} \right), \mathcal{L} \left( \frac{\partial}{\partial x^j} \right) \right] = 0, \]
whence

3. \( C^1(\mathfrak{A}_n) \simeq \mathbb{R} \left[ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right] \{ \theta^1, \ldots, \theta^n \} \)
i.e. is a free module with \( n \) generators over the polynomial ring in \( n \) generators.

**Proposition 4.3.** We have following identities in \( C^\bullet(\mathfrak{A}_n) \)

1. \( d\theta^i + \sum_j \theta^j \wedge \theta^i = 0, \)

2. \( d\theta^i_k + \sum_j \left( \theta^j_{ik} \wedge \theta^j + \theta^j_k \wedge \theta^i_k \right) = 0, \)

3. \( d\theta^i_{kl} + \sum_j \left( \theta^i_{kl} \wedge \theta^j + \theta^j_{ik} \wedge \theta^j + \theta^j_{il} \wedge \theta^i_k + \theta^i_k \wedge \theta^i_l \right) = 0. \)
Proof.

\[ d\theta^i(X, Y) = X\theta^i(Y) - Y\theta^i(X) - \theta^i([X, Y]) = -\theta^i([X, Y]), \]

where \( X = \sum_j f_j \frac{\partial}{\partial x^j}, \) \( X = \sum_j g^k \frac{\partial}{\partial x^k}. \)

\[ [X, Y] = \sum_{j,k} \left( f_j \frac{\partial g^k}{\partial x^j} - g^k \frac{\partial f_j}{\partial x^k} \right) = \]

\[ = \sum_k \left( \sum_j \left( f_j \frac{\partial g^k}{\partial x^j} - g^k \frac{\partial f_j}{\partial x^k} \right) \right) \frac{\partial}{\partial x^k}. \]

Hence

\[ d\theta^i(X, Y) = \sum_j \left( f_j \frac{\partial g^i}{\partial x^j} - g^i \frac{\partial f_j}{\partial x^j} \right) \frac{\partial}{\partial x^j}. \]

On the other hand

\[ \theta^i_j \wedge \theta^j(X, Y) = \theta^i_j(X)\theta^j(Y) - \theta^i_j(Y)\theta^j(X) = \]

\[ = \sum_j \left( -\frac{\partial f^i_j}{\partial x^j} + \frac{\partial g^i_j}{\partial x^j} f^i \right). \]

This proves (1). To obtain (2) we apply \( \mathcal{L} \left( \frac{\partial}{\partial x_k} \right) \), and applying \( \mathcal{L} \left( \frac{\partial}{\partial x_l} \right) \) to (2) we obtain (3) etc. These equations completely determine differential \( d \). 

Denote

\[ R^i_j := d\theta^i_j + \sum_k \theta^i_j \wedge \theta^k \in \mathcal{C}^2(\mathfrak{A}_n) = \Lambda^2 \mathfrak{A}_n^*. \]

Then equation (2) becomes

\[ 2' \]

\[ R^i_j = -\sum_k \theta^i_{jk} \wedge \theta^k. \]

Proposition 4.4. 1.

\[ R^i_j \wedge \theta^j = 0, \]

2.

\[ dR^i_j = \sum_k \left( R^i_k \wedge \theta^k - \theta^i_k \wedge R^j_k \right). \]

Proof. From (2')

\[ R^i_j \wedge \theta^j = -\sum_k \theta^i_j \wedge \theta^k \wedge \theta^j = 0 \]

since \( \theta^i_j = \theta^k_{kj}. \)

From (2)

\[ dR^i_j = \sum_k \left( d\theta^i_k \wedge \theta^j_k - \theta^i_k \wedge d\theta^j_k \right) = \]

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\[
\sum_k \left( - \sum_l (\theta_{ik}^l \wedge \theta_{il}^k + \theta_{ik}^l \wedge \theta_{il}^k) \wedge \theta_{il}^k + \sum_l (\theta_{lj}^k \wedge \theta_{lk}^l + \theta_{lj}^k \wedge \theta_{lk}^l) \wedge \theta_{lk}^l \right) = \\
= \sum_{k,l} \left( R_{lj}^k \wedge \theta_{lk}^l - \theta_{ik}^l \wedge \theta_{ik}^l \wedge \theta_{lk}^l + \theta_{ik}^l \wedge R_{lj}^k \wedge \theta_{lk}^l \right) = \\
= \sum_k \left( R_{lj}^k \wedge \theta_{lk}^l - \theta_{ik}^l \wedge R_{lj}^k \right).
\]

Corollary 4.5. The subalgebra \( \overline{W}_n := \mathbb{R}\{\theta_{ij}, R_{ij}\} \) is closed under \( d \) and finite dimensional.

**Proof.** Finite dimension follows from \((2')\). \(\square\)

### 4.3 Some ”soft” results

We describe the grading on an algebra \( \mathfrak{A}_n \).

\( \mathfrak{A}_n = \{ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i(x) = \sum_{\alpha} c^i_{\alpha} x^\alpha \in \mathbb{R}[[x_1, \ldots, x_n]], \alpha = (\alpha_1, \ldots, \alpha_n) \} \).

\( \mathfrak{A}_n = \mathbb{R}^n \oplus \mathfrak{gl}_n(\mathbb{R}) \oplus \ldots \)

One has

\([x^i \frac{\partial}{\partial x^j}, x^k \frac{\partial}{\partial x^l}] = \delta^l_j x^k \frac{\partial}{\partial x^i} - \delta^k_i x^l \frac{\partial}{\partial x^j} \).

To see grading we take \( E = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} \in \mathfrak{A}_n \). Then

\([E, X] = \sum_j \sum_i \left( x^i \frac{\partial f^j}{\partial x^i} - f^j \right) \frac{\partial}{\partial x^j} \)

and if \( f^j = c^j_{\alpha_1} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \) with \(|\alpha| = r\), then

\([E, c^j_{\alpha} x^\alpha \frac{\partial}{\partial x^j}] = \sum_i x^i \frac{\partial}{\partial x^i}, c^j_{\alpha} x^\alpha \frac{\partial}{\partial x^j} = \\
= \sum_i \alpha_i x^\alpha \frac{\partial}{\partial x^j} - \sum_i x^\alpha \delta^i j \frac{\partial}{\partial x^i} = (|\alpha| - 1)x^\alpha \frac{\partial}{\partial x^j} \).

Thus each monomial is an eigenvector for \( E \), and we can write \( \mathfrak{A}_n \) as a sum of eigenspaces

\( \mathcal{L}_E(x^\alpha \frac{\partial}{\partial x^j}) = (|\alpha| - 1)x^\alpha \frac{\partial}{\partial x^j} \),

\( \mathfrak{A}_n^{(p)} := \{ X \in \mathfrak{A}_n \mid \mathcal{L}_E(X) = pX \} \),

\( \mathfrak{A}_n = \bigoplus_{p=-1}^\infty \mathfrak{A}_n^{(p)}, \quad E|\mathfrak{A}_n^{(p)} = p \cdot \text{Id} \).

It is a grading, i.e.

\([\mathfrak{A}_n^{(p)}, \mathfrak{A}_n^{(q)}] \subset \mathfrak{A}_n^{(p+q)} \).
We have a dual grading on the Gelfand-Fuks complex $C^\bullet(\mathfrak{A}_n) = \Lambda^\bullet\mathfrak{A}_n^*$. One has the Lie derivative

$$L_E : \mathfrak{A}_n^* \to \mathfrak{A}_n^*.$$  

The dual grading on $\mathfrak{A}_n^*$ can be described as

$$(\mathfrak{A}_n^*)^{(p)} := \{ \omega \in \mathfrak{A}_n^* \mid L_E(\omega) = -p\omega \}.$$  

This induces a grading on G-F complex

$$C_m(\mathfrak{A}_n^{(p)}) = (\bigoplus \Lambda^{k-1}(\mathfrak{A}_n^*)^{(-1)} \otimes \Lambda^{k_0}(\mathfrak{A}_n^*)^{(0)} \otimes \ldots \otimes \Lambda^{k_r}(\mathfrak{A}_n^*)^{(r)}),$$

where

$$k_{-1} + k_0 + \ldots = m, \quad -k_{-1} + k_1 + 2k_2 + \ldots + rk_r = p.$$  

We have $L_E d = d L_E$ (so $L_E$ is a map of complexes). We can restrict to degree $p$

$$L_E\big|_{C^\bullet(\mathfrak{A}_n^{(p)})} = -p \cdot \text{Id}$$

**Proposition 4.6.**

\[ \dim H^*_{GF}(\mathfrak{A}_n) < \infty, \forall n \geq 0, \]

\[ H^m_{GF}(\mathfrak{A}_n) = 0, \forall m > n^2 + 2n. \]

**Proof.** One has

$$L_E(\omega) = d L_E(\omega) + i_E d\omega$$

so any $\omega \in C^m(\mathfrak{A}_n^{(p)})$ with $p \neq 0$ such that $d\omega = 0$ is exact, since then

$$d L_E(\omega) = L_E(\omega) = -p\omega.$$  

This gives on cohomology

$$H^m_{GF}(\mathfrak{A}_n) = H^m_{GF}(\mathfrak{A}_n) =: H^m(C^\bullet(\mathfrak{A}_n^{(0)}),$$

where

$$C^m(\mathfrak{A}_n^{(0)}) = (\Lambda^m\mathfrak{A}_n^*)^{(0)} = \bigoplus \Lambda^{k-1}(\mathfrak{A}_n^*)^{(-1)} \otimes \Lambda^{k_0}(\mathfrak{A}_n^*)^{(0)} \otimes \ldots \otimes \Lambda^{k_r}(\mathfrak{A}_n^*)^{(r)},$$

$$-k_{-1} + k_1 + 2k_2 + \ldots + rk_r = 0, \quad k_{-1} + k_0 + k_1 + \ldots + k_r = m.$$  

Since

$$\dim \mathfrak{A}_n^{(-1)} = \dim \mathbb{R}^n = n \implies k_{-1} \leq n,$$

$$\dim \mathfrak{A}_n^{(0)} = n^2 \implies k_0 \leq n^2.$$  

Furthermore

$$k_1 \leq n, k_2 \leq \frac{n}{2}, \ldots, k_n \leq 1.$$  

Hence

$$\dim C^m(\mathfrak{A}_n^{(0)}) < \infty \text{ for } m \geq 0,$$

$$C^m(\mathfrak{A}_n^{(0)}) = 0 \text{ for } m > n^2 + 2n.$$  

\[ \square \]
Example 4.7. For \( n = 1 \) we have following

\[
k_1 + 2k_2 + \ldots + k_r = k_{-1},
\]

\[
k_{-1} + k_0 + k_1 + \ldots + k_r \leq 3.
\]

This gives

\[
k_1 \leq 1, k_2 \leq \frac{1}{2} \text{ etc.} \implies k_2 = \ldots = k_r = 0.
\]

The dual algebra

\[
\mathfrak{A}_n^* \cong \bigoplus_{\text{deg}=\text{-}1} \mathbb{R}\theta_1^1 \oplus \bigoplus_{\text{deg}=0} \mathbb{R}\theta_1^1 \oplus \bigoplus_{\text{deg}=1} \mathbb{R}\theta_1^1 \oplus \ldots
\]

If \( k_{-1} = 0 \) then \( k_1 = k_2 = \ldots = 0 \) hence the only one allowed is

\[
\Lambda^*(\mathfrak{A}^*_1)^{(0)} = \mathbb{R} \oplus \mathbb{R}\theta_1^1.
\]

For \( k_{-1} = 1 \) we have \( k_1 = 1 \) and

\[
\Lambda^1(\mathfrak{A}^*_1)^{(-1)} \otimes \Lambda^*(\mathfrak{A}^*_1)^{(0)} \otimes \Lambda^1(\mathfrak{A}^*_1)^{(1)}
\]

Thus we need only to look at the subcomplex

\[
\mathbb{R}\{1, \theta_1^1, \theta^1 \wedge \theta_1^{11}, \theta^1 \wedge \theta_1^1 \wedge \theta_1^{11}\}
\]

because \( R^1_1 = d\theta_1^1 = -\theta_1^{11} \wedge \theta^1 \neq 0 \), so the cohomology is

\[
H^*_\text{GF} = \bigoplus_{\text{dim}=0} \mathbb{R} \bigoplus \mathbb{R}(\theta_1^1 \wedge R^1_1) \bigoplus \mathbb{R}(\theta_1^1 \wedge R^1_1) \bigoplus \mathbb{R}(\theta_1^1 \wedge R^1_1)
\]

4.4 Spectral sequences

The algebra generated by \( \{\theta_j^i, R_j^i\} \) is closed under the differential \( d \), so we have a subcomplex

\[
(\mathbb{R}\{\theta_j^i, R_j^i\}, d) =: (\overline{W}_n, d) \subset (\mathcal{C}^*(\mathfrak{A}_n), d).
\]

where

\[
\mathbb{R}\{\theta_j^i, R_j^i\} \cong \Lambda^*\mathfrak{gl}_n(\mathbb{R})^* \otimes S_n(\mathfrak{gl}_n(\mathbb{R})^*)
\]

Theorem 4.8. The inclusion

\[
(\overline{W}_n, d) \hookrightarrow (\mathcal{C}^*(\mathfrak{A}_n), d)
\]

is a quasi-isomorphism (induces isomorphism on cohomology).

The proof uses Hochschild-Serre spectral sequence, which we describe next.
4.4.1 Exact couples

Assume we have an exact sequence of the form

\[ A \xrightarrow{i} A \xrightarrow{j} B \xrightarrow{k} \]

It is called an exact couple. Define

\[ d: B \to B, \quad d := jk, \quad d^2 = jkjk = 0, \quad \text{and} \]

\[ H(B) := \ker d/\operatorname{im} d. \]

Now we can form derived couple taking

\[ A' \xrightarrow{i'} A' \xrightarrow{j'} B' = H(B) \]

where

- \( A' := i(A) \),
- \( B' := H(B) \),
- \( i'(a') = i(a') = i(i(a)) \),
- \( j'(a') = [j(a)] \) for \( a' = i(a) \),
- \( k'(b) = k(b) \).

Check this definitions for independence of representatives. The derived couple is again exact couple.

4.4.2 Filtered complexes

Let \((C^\bullet, d)\) be a filtered complex i.e. there is a sequence of subcomplexes

\[ C^\bullet = C_0^\bullet \supset C_1^\bullet \supset C_2^\bullet \supset \ldots \]

Let

\[ A := \bigoplus_{p \in \mathbb{Z}} C_p, \quad B := \bigoplus_{p \in \mathbb{Z}} C_p/C_{p+1} \]

Inclusions \( C_{p+1} \hookrightarrow C_p \) induce exact sequence

\[ 0 \to A \xrightarrow{i} A \xrightarrow{B} 0, \]

a long exact sequence of homology

\[ \ldots H(A) \xrightarrow{i_*} H(A) \xrightarrow{j_*} H(B) \xrightarrow{k_*} A \to \ldots, \]
and an exact couple

\[ A_1 := H(A) \xrightarrow{j_*} H(A) \]

\[ \xrightarrow{j_*} H(B) =: B_1 \]

### 4.4.3 Illustration of convergence

Consider simple case, filtration of a complex \( H(C^\bullet) \)

\[
\ldots = C_{-2} = C_{-1} = C_0 \supset C_1 \supset C_2 \supset 0 = \ldots
\]

\[
\ldots = C_{-2} = C_{-1} = C_0 \supset C_1 \supset C_2 = 0 = \ldots
\]

\[
\ldots = C_{-2} = C_{-1} = C_0 \supset C_1 \supset C_2 = 0 = \ldots
\]

Here

\[
B = \ldots \oplus 0 \oplus 0 \oplus C_0/\text{ker } \iota \oplus C_1/\text{ker } \iota \oplus C_2/\text{ker } \iota \oplus 0 \oplus \ldots
\]

Taking homology we get sequences

\[
H(C^\bullet) = H(C_0) \leftarrow H(C_1) \leftarrow H(C_2) \leftarrow 0 \leftarrow \ldots
\]

\[
A_1 := \bigoplus_{p \in \mathbb{Z}} H(C_p)
\]

\[
H(C^\bullet) = H(C_0) \supset \iota_* H(C_1) \leftarrow \iota_* H(C_2) \leftarrow 0 \leftarrow \ldots
\]

\[
A_2 := \bigoplus_{p \in \mathbb{Z}} \iota_* H(C_p)
\]

\[
H(C^\bullet) = H(C_0) \supset \iota_* H(C_1) \supset \iota_* \iota_* H(C_2) \leftarrow 0 \leftarrow \ldots
\]

\[
A_3 := \bigoplus_{p \in \mathbb{Z}} \iota_* \iota_* H(C_p).
\]

When we reach the stage in which all maps become inclusions, process is stationary i.e.

\[
A_3 = A_4 = \ldots
\]

\[
A_3 \xleftarrow{i} A_3
\]

\[
\xleftarrow{j} B_3 = H(A_3)
\]

where \( i \) is inclusion, \( \text{im } k = \ker i = 0 \) so \( k = 0 \). This means that also

\[
B_3 = B_4 = \ldots
\]

since \( d = kj = 0 \)
4.4.4 Hochschild-Serre spectral sequence

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of a Lie algebra $\mathfrak{g}$.

\[
C^\bullet(\mathfrak{g}; M) = \text{Hom}(\Lambda^\bullet \mathfrak{g}, M), \quad d: C^\bullet(\mathfrak{g}; M) \to C^{\bullet+1}(\mathfrak{g}; M)
\]

\[
d\omega(X_0, X_1, \ldots, X_r) = \sum_i (-1)^i X_i \omega(X_0, \ldots, \hat{X}_i, \ldots, X_r) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r).
\]

Define the filtration on the above complex by

\[
F_p C^{p+q}(\mathfrak{g}; M) := \{ \omega \in C^{p+q} | \iota_{X_0} \ldots \iota_{X_q} \omega = 0 \forall X_1, \ldots, X_q \in \mathfrak{h} \}.
\]

This means that we can associate with $\omega \in F_p C^{p+q}$ an element $\phi(\omega) \in \text{Hom}(\Lambda^q \mathfrak{h}, \text{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}); M))$ given by the formula

\[
\phi(\omega)(X_1, \ldots, X_q)(\overline{Y_1}, \ldots, \overline{Y_p}) = \omega(X_1, \ldots, X_q, Y_1, \ldots, Y_p).
\]

Then

\[
\ker \phi = F^{p+1} C^{p+q},
\]

Hence there is a spectral sequence with

\[
E_0^{p,q} \simeq C^q(\mathfrak{h}; \text{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}); M)), \quad d_0 = d,
\]

\[
E_1^{p,q} \simeq H^q(\mathfrak{h}; \text{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}); M)),
\]

\[
E_2^{p,0} \simeq H^p(\mathfrak{g}, \mathfrak{h}; M),
\]

\[
E_\infty \Rightarrow H^*(\mathfrak{g}; M)
\]

Now we are ready to prove that the inclusion

\[
i: \tilde{W}_n \to C^\bullet(\mathfrak{A}_n)
\]

induces an isomorphism

\[
H^*(\tilde{W}_n, d) \simeq H^*_G(\mathfrak{A}_n)
\]

that is theorem (4.8).

Proof. Both $\tilde{W}_n$ and $C^\bullet(\mathfrak{A}_n)$ are filtered differential graded algebras, and their associated spectral sequences converge to $H^*(\tilde{W}_n)$ and respectively to $H^*_G(\mathfrak{A}_n)$. On the other hand $i$ induces isomorphism on the level of $E_1$.

First $\tilde{W}_n$ is graded by

\[
\tilde{W}_n^{-p} = \bigoplus_{r+2s=p} \Lambda^r(\tilde{\theta}_j^i) \otimes S^n[R_j^i]
\]

and then

\[
F^p \tilde{W}_n^{-p+q} := \{ \omega \in \tilde{W}_n^{-p+q} | \iota_{X_0} \ldots \iota_{X_q} \omega = 0 \forall X_0, \ldots, X_q \in \mathfrak{A}_n(0) \}
\]

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Fact 4.9.

\[
E^p,q_0 \simeq \begin{cases} 0, & p \text{ odd or } p > 2n, \\
C^q(\mathfrak{A}_n^{(0)}; S^p_n[R_j^i]), & p \text{ even and } p \leq 2n.
\end{cases}
\]

\[
E^p,q_1 \simeq \begin{cases} 0, & p \text{ odd or } p > 2n, \\
H^q_{GF}(\mathfrak{A}_n^{(0)}; S^p_n[R_j^i]), & p \text{ even and } p \leq 2n.
\end{cases}
\]

The filtration on \( C^\bullet(\mathfrak{A}_n) = \bigoplus_p C^p(\mathfrak{A}_n) \) is the Hochschild-Serre filtration relative to \( \mathfrak{A}_n^{(0)}. \)

Fact 4.10.

\[
E^p,q_1 \simeq H^q_{GF}(\mathfrak{A}_n^{(0)}; F^pC^p(\mathfrak{A}_n)).
\]

It is a filtration, so 

\[
[\mathfrak{A}_n^{(0)}, \mathfrak{A}_n^{(p)}] \subset \mathfrak{A}_n^{(p)}
\]

and we have an action of \( \mathfrak{g}t_n(\mathbb{R}) = \mathfrak{A}_n^{(0)} \) on \( \mathfrak{A}_n^{(p)} \) for each \( p. \) Since \( \mathfrak{A}_n^{(0)} \) acts semisimply on the coefficients one gets further

\[
E^p,q_1 \simeq H^q_{GF}(\mathfrak{A}_n^{(0)}; \Lambda^p(\mathfrak{A}_n^{(0)}))^*) \simeq H^q_{GF}(\mathfrak{A}_n^{(0)}; B^p),
\]

where

\[
B^p := \{ \omega \in C^p(\mathfrak{A}_n) \mid \iota_X \omega = 0 = \mathcal{L}_X \omega \forall X \in \mathfrak{A}_n^{(0)} \}
\]

are the basic elements with respect to \( \mathfrak{A}_n^{(0)}. \) Note that if \( Y = Y^r_s = X^r \partial_{x^s} \)

\[
\iota_Y R^i_j = -\iota_Y (\theta^i_j k \wedge \theta^k) = 0,
\]

whence the map

\[
E^p,q_1(\tilde{W}_n) \rightarrow E^p,q_1(C^\bullet(\mathfrak{A}_n)).
\]

Lemma 4.11. The inclusion \( \iota: \tilde{W}_n \hookrightarrow C^\bullet(\mathfrak{A}_n) \) induces an isomorphism between the \( \mathfrak{A}_n^{(0)} \)-basic elements of \( \tilde{W}_n \) and \( C^\bullet(\mathfrak{A}_n) \).

Proof. Elementary invariance theory to eliminate the form \( \theta^\alpha \) with \( |\alpha| > 2. \)

Again let

\[
W_n = \Lambda(u_1, \ldots, u_n) \otimes S_n[c_1, \ldots, c_n]
\]

\[
\deg(u_i) = 2i - 1, \quad \deg(c_i) = 2i, \quad du_i = c_i, \quad dc_i = 0.
\]

\[
\tilde{W}_n = \Lambda(\theta^i_j) \otimes S_n[R^i_j]
\]

Proposition 4.12. The map

\[
c_i \mapsto c_i(R), \quad R = (R^i_j)
\]

has an extension to a map of complexes \( W_n \rightarrow \tilde{W}_n. \) Any such extension induces isomorphism in cohomology

\[
H^*(W_n) \cong H^*(\tilde{W}_n).
\]
For example if $n = 1$ we have

$$c_1 \mapsto c_1(R) = R_1^1,$$

$$u_1 \mapsto \theta_1^1.$$

**Proof.**

$$E_1^{0,2q-1}(\overline{W_n}) = H^{2q-1}(\mathfrak{gl}_n(R); \mathbb{R}) \ni u_j,$$

where $u_j$ is a generator for $j = 1, \ldots, n$. Now each $u_j$ has a representative $[w_j]$ such that

$$w_j \in F^0\overline{W_n}^{2q-1}, \quad dw_j = c_j \in F^2\overline{W_n}^{2q-1}$$

thus giving a basic element of $\overline{W_n}$ in

$$E_1^{2q,0} \simeq S^q(R_j^{\text{inv}}).$$

The basic elements of $\overline{W_n}$ form an algebra isomorphic to $\mathbb{R}[c_1, \ldots, c_n]$.

The extension is given by

$$u_j \mapsto w_j,$$

$$c_j \mapsto dw_j.$$

Filtering $W_n$ by the ideals $F^pW_n$ generated by polynomials of degree at least $p$ in the $c_i$'s one obtains a morphism of complexes compatible with filtrations, which induces isomorphism on the level of $E_1$. 

In the relative case $\mathfrak{o}_n \subset \mathfrak{gl}_n(R) = \mathfrak{sl}_n^{(0)}$ gives actions of $\mathfrak{o}_n$ on $\overline{W_n}$ and $C^\bullet(\mathfrak{A}_n)$. Passing to the subalgebras of $\mathfrak{o}_n$-basic elements, then restricting the filtrations one obtains isomorphisms

$$H^*(WO_n) \simeq H^*(\overline{W_n}, \mathfrak{o}_n) \simeq H^*_GF(\mathfrak{A}_n, \mathfrak{o}_n),$$

where

$$WO_n = \Lambda(u_1, u_3, \ldots, u_k) \otimes S_n[c_1, \ldots, c_n],$$

$$du_{2j-1} = c_{2j}, \quad dc_j = 0.$$

**Corollary 4.13.** Any class in $H^*(\mathfrak{A}_n)$ (respectively $H^*(\mathfrak{A}_n, \mathfrak{o}_n)$) has a representative which depends only on the second jet.
Chapter 5

Characteristic maps and Gelfand-Fuks cohomology

5.1 Jet groups

Definition 5.1. Let \( x \in \mathbb{R}^n \) and let \( f : U \to \mathbb{R}^n \) be a \( C^\infty \)-function. Then \( j^k_x(f) \) is an equivalence class with respect to

\[
f \sim_k g \iff \frac{\partial^{\alpha} f}{\partial x^\alpha} \bigg|_x = \frac{\partial^{\alpha} g}{\partial x^\alpha} \bigg|_x, \quad \forall \alpha = \alpha_1 + \ldots + \alpha_n \leq k.
\]

Then \( \mathcal{G}_k(n) := \{ j^k_0(f) \mid f \text{ local diffeomorphism of } \mathbb{R}^n, \ f(0) = 0 \} \)

is a Lie group under composition

\[
j^k_0(f) \circ j^k_0(g) := j^k_0(f \circ g).
\]

Identifying with polynomial representatives

\[
j^k_0(f) \simeq \{ \sum_{1 \leq |\alpha| \leq k} a^j_\alpha x^\alpha \in \mathcal{P}_0^k[x_1, \ldots, x_n] \mid 1 \leq j \leq n \}
\]

Then \( j^k_0(f) \in \mathcal{G}_k(n) \) means \( a^j_\alpha \in \text{GL}_n(\mathbb{R}) \).

One has a sequence of projections

\[ G_\infty(n) := \ldots \to G_{k+1}(n) \to G_k(n) \to \ldots \to G_1(n). \]

If \( h = f \circ g \)

\[ h^i(x^1, \ldots, x^n) = f^i(g^1(x^1, \ldots, x^n), \ldots, g^n(x^1, \ldots, x^n)) \]

\[ c^i_k := \frac{\partial h^i}{\partial x^k} \bigg|_0 = \sum_l \frac{\partial f^i}{\partial x^l} \bigg|_0 \frac{\partial g^l}{\partial x^k} \bigg|_0 = \sum_l a^j_l b^i_k. \]

\[ c^i_{jk} := \frac{\partial^2 h^i}{\partial x^j \partial x^k} \bigg|_0 = \sum_{l,s} \frac{\partial^2 f^i}{\partial x^s \partial x^l} \bigg|_0 \frac{\partial g^s}{\partial x^j} \bigg|_0 + \sum_l \frac{\partial f^i}{\partial x^l} \bigg|_0 \frac{\partial^2 g^l}{\partial x^j \partial x^k} \bigg|_0 \]

so

\[ c^i_{jk} = \sum_{l,s} a^j_l b^s_k + \sum_l a^j_l b^i_{jk} \]
etc. In particular \( \ker(G_2(n) \to G_1(n)) \) has multiplication

\[
c^i_{jk} = a^i_{jk} + b^i_{jk}.
\]

In general

\[
N_k(n) := \ker(G_k(n) \to G_1(n))
\]

is a vector space equipped with a polynomial multiplication which implies that \( N_k(n) \) is a nilpotent Lie subgroup, and

\[
G_k(n) = G_1(n) \rtimes N_k(n).
\]

\[
g_k(n) := \text{Lie}(G_k(n)) \simeq \{ j^k_0 X \mid X = \sum_i \frac{\partial}{\partial x^i}, \ X(0) = 0 \}
\]

with the bracket

\[
[j^k_0(X), j^k_0(Y)] = -j^k_0([X,Y]).
\]

5.2 Jet bundles

**Definition 5.2.** Let \( M^n \) be a \( C^\infty \)-manifold. The jet bundle on \( M \)

\[
J^k(M) := \{ j^k_0(f) \mid f : U \subset \mathbb{R}^n \to M \text{ local diffeomorphism at } 0 \in U \}.
\]

It has a tautological \( C^\infty \)-structure modelled on

\[
J^k(\mathbb{R}^n) = \mathcal{P}_k(n) \simeq \text{polynomial jets}
\]

Again one has a sequence of natural projections

\[
J^\infty(M) := \ldots \to J^{k+1}(M) \to J^k(M) \to \ldots \to J^1(M) \to M,
\]

which are principal bundles with structure groups

\[
G_\infty(n) := \ldots \to G_{k+1}(n) \to G_k(n) \to \ldots \to G_1(n).
\]

\( J^1(M) = F(M) \to M \) is a frame bundle with the structure group \( \text{GL}_n(\mathbb{R}) = G_1(n) \).

There is a natural (commuting with \( \text{Diff}_M \)) map

\[
\mathfrak{A}_n \xrightarrow{\sim} T_{j^\infty_0(\phi)} J^\infty(M)
\]

For

\[
X \in \mathfrak{A}_n, \ X = \sum_i f^i \frac{\partial}{\partial x^i}
\]

and a 1-parameter family \( \psi_t \) of local diffeomorphism of \( \mathbb{R}^n \) such that

\[
\psi_t(0) = 0, \ \psi_0 = \text{Id}, \ X = j^\infty_0 \left( \frac{d\psi_t}{dt} \bigg|_{t=0} \right),
\]

we have a curve in a manifold of jets \( j^\infty_0(\psi_t) \). For a local diffeomorphism \( \phi : \mathbb{R}^q \to M^n \) we have a curve passing through \( \phi \)

\[
j^\infty_0 \left( \frac{d}{dt}(\phi \circ \psi_t) \bigg|_{t=0} \right)
\]

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and
\[ X = \frac{d}{dt} j_0^{\infty}(\psi_t) \big|_{t=0} = j_0^{\infty} \left( \frac{d\psi_t}{dt} \big|_{t=0} \right). \]

Let \( u = j_0^{\infty}(\phi) \in J^\infty(M) \), and define
\[
\tilde{X}_u := j_0^{\infty} \left( \frac{d}{dt} \phi \circ \psi_t \big|_{t=0} \right) = \frac{d}{dt} (\phi \circ \psi_t) \big|_{t=0} \in T_u J^\infty(M), \quad \phi \circ \psi_t \big|_{t=0} = \phi.
\]

The map
\[
\mathfrak{A}_n \to T_u J^\infty(M), \quad X \mapsto \tilde{X}_u
\]
is natural i.e. it commutes with the action of the diffeomorphisms
\[
T_{j_0^{\infty}(\rho \circ \phi)} J^\infty(M) \quad \rho^* \quad T_{j_0^{\infty}(\phi)} J^\infty(M)
\]

**Proposition 5.3.** We have a natural isomorphism of differential graded algebras
\[
(C^* (\mathfrak{A}_n), d) \simeq (\Omega^* (J^\infty(M))^{\text{Diff}} M, -d).
\]

**Proof.** We take for \( u = j_0^{\infty}(\phi) \)
\[
\tilde{\omega}_u(\tilde{X}_u^1, \ldots, \tilde{X}_u^p) := \omega(X_1, \ldots, X^p).
\]

In particular if we set for a basis \( \{ \theta^i \} \) of \( \mathfrak{A}^*_n \)
\[
\tilde{\theta}^i_u(\tilde{X}_u) = \frac{\partial^{|\alpha|} f^i}{\partial x^\alpha} \big|_{x=0} = (-1)^{|\alpha|} \theta^i_u(X)
\]
then they satisfy the same differential equations as \( \theta^i_u \).

**Example 5.4.** In local coordinates \( (v_1, \ldots, v_n) \) around \( u = j_0^{\infty}(\phi) \)
\[
\left\{ v_i \big|_u, v^j \right\} := \left\{ \frac{\partial (v^i \circ \phi)}{\partial x^j} \big|_u, v^{i j}_k \right\} := \frac{\partial^2 (v^i \circ \phi)}{\partial x^j \partial x^k} \big|_u, \ldots, v_{\alpha} = \frac{\partial^{|\alpha|} (v^i \circ \phi)}{\partial x^\alpha} \big|_u
\]
on one has
\[
dv_{\alpha} = \sum_{\beta + \gamma = \alpha} v^i_{\beta[k]} \tilde{\theta}^{k}_{\gamma}, \quad \beta[k] := (\beta_1, \ldots, \beta_k + 1, \ldots, \beta_n).
\]

### 5.3 Characteristic map for foliation

Let \((M, \mathcal{F})\) be a manifold with foliation, which we can describe by a 1-cycle with values in \( \Gamma_q \) given by the following data

1. an open cover \( M = \bigcup_\alpha U_\alpha \),
2. \( \forall \alpha \) there is a submersion \( f_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^q \)
3. ∀x ∈ Uα ∩ Uβ there is a local diffeomorphism gαβ : Vα → Vβ (neighbourhoods of fα(x) and fα(x) respectively) such that fβ = gαβ ◦ fα near x.

Then

\[ f_\alpha^*(J^\infty(V_\alpha)) \to U_\alpha, \text{ and } f_\beta^*(J^\infty(V_\beta)) \to U_\beta \]

can be identified over Uα ∩ Uβ via j_{\beta\alpha}^\infty(g_{\beta\alpha}), giving the principal G^k(q)-bundles over M:

\[ J^\infty(\mathcal{F}) := \ldots \to J^{k+1}(\mathcal{F}) \to J^k(\mathcal{F}) \to \ldots \to J^2(\mathcal{F}) \to J^1(\mathcal{F}) \to M. \]

This are jet bundles of "transverse local diffeomorphisms". In particular J^1(\mathcal{F}) is a principal GL_q(\mathbb{R})-bundle associated to the transverse bundle Q(\mathcal{F}) = TM/\mathcal{F} - bundle of transverse frames.

The forms θ_i on J^\infty(V_\alpha) are invariant under Diff hence they also define forms on J^\infty(\mathcal{F}). They are the "canonical forms" on J^\infty(\mathcal{F}).

The characteristic homomorphisms

\[ \chi_{GF} : C^\bullet(\mathfrak{A}_q) \to \Omega^\bullet(J^\infty(\mathcal{F})) \]

is defined by sending ω to the lift to M of the Diff-invariant forms \tilde{ω}_\alpha on V_\alpha. It is a homomorphism of DGA’s inducing

\[ \chi_{GF}^* : H^*_GF(\mathfrak{A}_q) \to H^*(J^\infty(\mathcal{F})) \simeq H^*(J^1(\mathcal{F})). \]

**Remark 5.5 (Bott’s vanishing theorem revisited).** Any E-flat (Bott) connection (def. (2.5)) \n\n\n∇^e on Q is given by a gl_n(\mathbb{R})-valued form on J^1(\mathcal{F}) which is of the form \omega_i^j = s^\ast(\tilde{\theta}_i^j) for some GL_n(\mathbb{R}) -equivariant section s : J^1(\mathcal{F}) \to J^2(\mathcal{F}). Then its curvature form

\[ \Omega^1_j = s^\ast(R^j_1) \implies \Omega^1_j \land \omega^j = s^\ast(R^j_1 \land \tilde{\theta}^j) = 0 \]

hence

\[ \Omega_{j_1}^1 \land \ldots \land \Omega_{j_p}^1 = 0, \ \forall p > q. \]

Assume the normal bundle Q = Q(\mathcal{F}) is trivializable and choose a global section s : M → \mathcal{F}. Then the diagram

\[ s^\ast \circ \chi_{GF}^* : H^*_GF(\mathfrak{A}_q) \longrightarrow H^*(M) \xrightarrow{\text{pr}^\ast} \Omega^*(J^1(\mathcal{F})) \]

is commutative.

Passing to the relative subcomplex one gets

\[ \chi_{GF}^{rel} : C^\bullet(\mathfrak{A}_q, O(n)) \to \Omega^\bullet(J^\infty/O(n)) \]

which induces

\[ \chi_{GF}^{rel} : H^*(\mathfrak{A}_q, O(n)) \to H^*(J^1(\mathcal{F})/O(n)) \simeq H^*(M). \]

The isomorphism

\[ \sigma^* : H^*(J^1(\mathcal{F})/O(n)) \to H^*(M) \]

is commutative.
is implemented by a metric on $Q$ (i.e. a section $\sigma: M \to J^1(\mathcal{F})/O(n)$). Then the diagram

$$
\begin{array}{c}
\text{H}^*(\mathfrak{A}_n, O(n)) \xrightarrow{\chi^\text{rel}_F} \text{H}^*(M) \\
\downarrow \chi^\Phi \\
\text{H}^*(WO_n)
\end{array}
$$

is again commutative.
Chapter 6

Index theory and noncommutative geometry

6.1 Classical index theorems

Let \((M, g)\) be a Riemannian manifold, \(g\)-metric. Index theorems describe properties of geometric elliptic operators in terms of topological characteristic classes.

For a selfadjoint elliptic operator \(D = D^*\)

\[
\text{Index}(D) := \dim \ker D - \dim \text{coker} D \in \mathbb{Z}
\]

We give a few examples of index theorems.

**Example 6.1.** Take a de Rham complex \(\Omega^\bullet(M)\) with

\[
d : \Omega^i(M) \to \Omega^{i+1}(M)
\]

and its adjoint

\[
d^* : \Omega^i(M) \to \Omega^{i-1}(M).
\]

One has even/odd grading on forms \((\gamma = (-1)^{\deg})\), and the operator

\[
d + d^* : \Omega^{ev} \to \Omega^{odd}
\]

is selfadjoint elliptic operator. Furthermore

\[
\text{Index}(d + d^*)^{ev} = \dim \ker (d + d^*)^{ev} - \dim \text{coker}(d + d^*)^{ev}
\]

and

\[
\ker (d + d^*) = H^*_dR(M; \mathbb{R}),
\]

\[
\ker (d + d^*)^{ev} = H^{ev}_dR(M; \mathbb{R}), \quad \text{coker}(d + d^*)^{odd} = H^{odd}_dR(M; \mathbb{R}).
\]

This means

\[
\text{Index}(d + d^*) = \dim H^{ev}(M; \mathbb{R}) - \dim H^{odd}(M; \mathbb{R}) = \chi(M)
\]

- the Euler characteristic of a manifold \(M\).

**Theorem 6.2 (Gauss-Bonnet).**

\[
\chi(M) = \text{Index}(d + d^*)^{ev} = \int_M \text{Pf}(R),
\]

where \(\text{Pf}(M)\) is a Pfaffian i.e. the square root of the determinant, and \(R\) - a curvature.
This theorem gives topological constraints on Gaussian curvature, for if \( n = 2 \) one has \( \text{Pf}(R) = K \). The right hand side depends on the metric, while on the left we have topological invariant.

**Example 6.3.** In the example above lets take different grading. Assume that \( \dim M = 4n \).

Take a Hodge star operator 

\[
* : \Omega^k(M) \to \Omega^{4n-k}.
\]

One has \( s^2 = (-1)^{k(4n-k)} \) so it gives rise to another grading \( \gamma \) on \( \Omega^*(M) \). It splits the complex into \( \Omega^-(M) \) and \( \Omega^+(M) \) (negative and positive eigenspaces). Furthermore

\[
\text{Index}(d + d^*)^+ = \dim H^{2n}(M)^+ - \dim H^{2n}(M) = \sigma(M)
\]

- the signature of \( M \) i.e. a signature of bilinear form

\[
H^{2n}(M) \times H^{2n}(M) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta.
\]

On the other side

**Theorem 6.4 (Hirzebruch signature thm.).**

\[
\sigma(M) = \text{Index}(d + d^*) = \int_M L(R), \quad L(R) := \left( \det \left( \frac{R}{2 \tanh \frac{R}{2}} \right) \right)
\]

as a formal series. \( L(R) \) is a \( L \)-genus of a manifold.

\( L(R) \) is a combination of Pontryagin classes which depends on a metric structure of a manifold.

**Example 6.5.** Let \( E \) be a holomorphic Hermitian bundle on a manifold \( M \). One has an operator \( \overline{\partial}_E \oplus \overline{\partial}_E^* \) on \( \Omega^0 \otimes S(E) \). Its index

\[
\text{Index}(\overline{\partial}_E \oplus \overline{\partial}_E^*) = \chi(E)
\]

- the Euler characteristic of a bundle \( E \). On the other hand

**Theorem 6.6 (Riemann-Roch-Hirzebruch).**

\[
\chi(E) = \text{Index}(\overline{\partial}_E \oplus \overline{\partial}_E^*) = \int_M \text{Td}(M) \text{ch}(E),
\]

where the Todd class of \( M \) and Chern character of \( E \) are given by

\[
\text{Td}(M) = \det \frac{R^{\text{hol}}}{e^{R^{\text{hol}}/2} - 1}, \quad \text{ch}(E) = \text{Tr}(e^{FE}).
\]

**Example 6.7.** The most general example one has for Dirac operator \( \slashed{D} \). One has a grading \( \slashed{D}^+, \slashed{D}^- \) from Spin-bundle.

\[
\text{Index} \slashed{D} = \dim \ker \slashed{D} - \dim \text{coker} \slashed{D} = S(M)
\]

- the spinor number of a manifold \( M \). On the other side

**Theorem 6.8 (Atiyah-Singer).**

\[
S(M) = \text{Index} \slashed{D} = \int_M \widehat{A}(R), \quad \widehat{A}(R) := (\det)^{1/2} \left( \frac{R}{2 \sinh \frac{R}{2}} \right)
\]

\( \widehat{A}(R) \) is another combination of Pontryagin classes. Together with Lichnerowicz theorem it gives constraints on scalar curvature.

Summarizing
6.2 General formulation and proto-index formula

Let $A$ be a $C^*$-algebra and $\mathfrak{A}$ its dense subalgebra such that if $a \in \mathfrak{A}$ has an inverse $a^{-1} \in A$, then $a^{-1} \in \mathfrak{A}$.

Example 6.9. $M$-closed manifold, $A = C(M)$, $\mathfrak{A} = C^\infty(M)$. Then

$$K^*(M) = K_*(C(M)) = K_*(C^\infty(M)),$$

(via Serre-Swan theorem) where the right hand side has algebraic definition (purely for $* = \text{even}$ and almost for $* = \text{odd}$).

In general

$$K_0(\mathfrak{A}) := \text{Idemp}(M_\infty(\mathfrak{A}))/\sim \cong \pi_1(\text{GL}_\infty(\mathfrak{A})),$$

where $\sim$ is some equivalence relation,

$$K_1(\mathfrak{A}) := \text{GL}_\infty(\mathfrak{A})/\text{GL}_\infty(\mathfrak{A})^0 \cong \pi_0(\text{GL}_\infty(\mathfrak{A})),$$

where $\text{GL}_\infty(\mathfrak{A})^0$ is a group of connected components. For the definition of $K_1(\mathfrak{A})$ we need a topology on $\mathfrak{A}$. We can replace $\text{GL}_\infty(\mathfrak{A})$ by $U_\infty(\mathfrak{A})$ (unitary matrices). From Bott periodicity $K_2(\mathfrak{A}) = K_0(A)$ and so on.

What is the dual (homology) theory? $K$-homology.

Assume $A \subset B(\mathcal{H})$ (bounded operators on Hilbert space $\mathcal{H}$). Let $F = F^* \in A$, Fredholm operator, such that

$$[F, A] \subset \mathcal{K}(\mathcal{H}), \text{ (compact operators)},$$

and moreover

$$[F, \mathfrak{A}] \subset \mathcal{L}^p(\mathcal{H}), \text{ (Schatten class)}$$

for some $p \geq 1$. The triple $(\mathfrak{A}, \mathcal{H}, F)$ is a $p$-summable Fredholm module. Together with grading $\gamma$ such that

$$\gamma^2 = \text{Id}, \quad \gamma = \gamma^*, \quad \gamma a = a \gamma \quad \forall \ a \in \mathfrak{A},$$

$$\gamma F + F \gamma = 0,$$

the quadruple $(\mathfrak{A}, \mathcal{H}, \gamma, F)$ is a K-cycle. The Hilbert space $\mathcal{H}$ decomposes into positive and negative eigenspaces of $\gamma$

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

and there is a decomposition of $F$

$$F = \begin{pmatrix} 0 & F^+ \\ F^- & 0 \end{pmatrix}.$$

**Lemma 6.10.** Let $F$ be bounded selfadjoint involution on $\mathcal{H}$ (i.e. $F^2 = \text{Id}$). Then

1. If $e^2 = e \in \mathfrak{A}$ then

$$F_e := eFe$$

is Fredholm operator.
2. If $g \in \text{GL}_1(\mathfrak{A})$ and $P = \frac{1+F}{2}$ then

$$F_g := PgP$$

is Fredholm operator.

Proof. Ad. 1

$$F_e^2 = eFeFe = e([F, e] + eF)e$$

which is a sum of $e$ and compact operator on $e\mathcal{H}e$.

Ad. 2

$$F_gF_g^{-1} = PgPg^{-1}P = Pg([P, g^{-1}] + g^{-1}P)P$$

which is a sum of $P$ and compact operator on $P\mathcal{H}P$.

If $e^2 = e \in M_N(\mathfrak{A}) = \mathfrak{A} \otimes M_N(\mathbb{C})$ then we can form

$$\mathcal{H}_N := \mathcal{H} \otimes \mathbb{C}^N, \quad F_N := F \otimes \text{Id}.$$ 

For an idempotent $e$, assignment

$$(F, e) \mapsto \text{Index}(F_e^+) \in \mathbb{Z}$$

extends to a pairing

$$K^0(\mathfrak{A}) \times K_0(\mathfrak{A}) \to \mathbb{Z}.$$ 

Similarly for $g \in \text{GL}_1(\mathfrak{A})$, assignment

$$(P, g) = \left(\frac{1+F}{2}, g\right) \mapsto \text{Index}(F_g) \in \mathbb{Z}$$

extends to a pairing

$$K^1(\mathfrak{A}) \times K_1(\mathfrak{A}) \to \mathbb{Z}.$$ 

Lemma 6.11 (Well known). Let $P, Q$ be bounded operators on a Hilbert space $\mathcal{H}$, such that

$$\text{Id} - QP, \text{Id} - PQ \in L^p.$$ 

Then $P, Q$ are Fredholm operatos and

$$\text{Index}(P) = \text{Tr}((\text{Id} - QP)^n) - \text{Tr}((\text{Id} - PQ)^n), \quad \forall \ n \geq p.$$ 

Proposition 6.12. Assume $[F, \mathfrak{A}] \in L^p$ (that is $(\mathfrak{A}, \mathcal{H}, F)$ is $p$-summable Fredholm module). Then

1. In the graded case, that is given $\gamma : \mathcal{H} \to \mathcal{H}$, one has for all projections $e$

$$\text{Index}(F_e^+) = (-1)^n \text{Tr}(\gamma e [F, e]^{2m}), \quad \forall \ 2m \geq p.$$ 

2. In the ungraded case one has for all $g \in \text{GL}_1(\mathfrak{A})$

$$\text{Index}(F_g) = \frac{1}{2^{2m+1}} \text{Tr}(g[F, g^{-1}]^{2m+1}), \quad \forall \ 2m \geq p.$$
Proof. In the graded case

\[ \text{Index}(F^+_e) = \text{Tr}(\gamma P_{\ker F_e}) = \text{Tr}(\gamma(e - F^2_e)^m) = \text{Tr}(\gamma(e - eFeF e)^m) \]

for \(2m = n \geq p\). Now as above


since

\[ [F, e] = [F, e^2] = [F, e]e + e[F, e]. \]

Thus

\[ \text{Tr}(\gamma(e - eFeF e)^m) = (-1)^m \text{Tr}(\gamma(e[F, e]^2)^m) = (-1)^m \text{Tr}(\gamma e([F, e])^2^m). \]

In the ungraded case one has

\[ \text{Index}(F_g) = \text{Tr}((P - P g^{-1} P gP)^m) - \text{Tr}((P - P g P g^{-1} P)^m) \]

for \(m\) sufficiently large. Furthermore

\[ P - P g^{-1} P g P = P + P([P, g^{-1}] - P g^{-1})g P = \]

\[ = P[P, g^{-1}]g P = -P[P, g^{-1}][(P, g) - P g) = \]

\[ = -P[P, g^{-1}][P, g] + P[P, g^{-1}]P g \]

because

\[ P^2 = P \implies [g^{-1}, P]P + P[g^{-1}, P] = [g^{-1}, P] \implies P[P, g^{-1}]P = 0. \]

Hence

\[ \text{Tr}((P - P g^{-1} P g P)^m) = (-1)^m \text{Tr}(P([P, g^{-1}][P, g])^m). \]

Writig again

\[ [P, g^{-1}] = P[P, g^{-1}] + [P, g^{-1}]P, \]

\[ [P, g] = P[P, g] + [P, g]P \]

one has

\[ P[P, g^{-1}][P, g] = P[P, g^{-1}][P, g]P = [P, g^{-1}][P, g]P. \]

Therefore

\[ \text{Tr}((P - P g^{-1} P g P)^m) = (-1)^m \text{Tr}(P([P, g^{-1}][P, g])^m) = \]

\[ = (-1)^m \text{Tr}\left( \frac{1 + F}{2} \left( \frac{1}{2} [F, g^{-1}]^2 \frac{1}{2} [F, g] \right)^m \right) = \]

\[ = \frac{(-1)^m}{2^{m+1}} \left( \text{Tr}(([F, g^{-1}][F, g])^m) + \text{Tr}(F([F, g^{-1}][F, g])^m) \right). \]

Changing \( g \) to \( g^{-1} \) one gets

\[ \text{Tr}((P - P g g^{-1} P)^m) = \frac{(-1)^m}{2^{m+1}} \left( \text{Tr}(([F, g][F, g^{-1}])^m) + \text{Tr}(F([F, g][F, g^{-1}])^m) \right). \]
Noting that
\[ [F, g^{-1}][F, g] = (-g^{-1}[F, g]g^{-1})(-g[F, g^{-1}]g) = g[F, g][F, g^{-1}]g \]
one has
\[ \text{Tr}(([F, g^{-1}][F, g])^m) = \text{Tr}(([F, g][F, g^{-1}])^m). \]
Now
\[ ([F, g^{-1}][F, g])^m = (-g^{-1}[F, g^{-1}]g)^m = (-1)^m (g^{-1}[F, g])^{2m}, \]
whence
\[ \text{Index}(F_g) = \frac{1}{2^{m+1}} \left( \text{Tr}(F[g^{-1}[F, g])^{2m}) - \text{Tr}(F[g][F, g^{-1}])^{2m}) \right). \]
The second term can be written as
\[ \text{Tr}(F[g^{-1}[F, g])^{2m}) = \text{Tr}(F(F, g)^{-1}[F, g])^{2m}) = \text{Tr}(F[g^{-1}[F, g])^{2m}) \]
So the difference gives
\[ \text{Index}(F_g) = \frac{1}{2^{m+1}} \text{Tr}((F - g^{-1}Fg)(g^{-1}[F, g])^{2m}) = \frac{1}{2^{m+1}} \text{Tr}((g^{-1}[F, g])^{2m+1}) = \frac{1}{2^{m+1}} \text{Tr}((g[F, g^{-1}])^{2m+1}). \]

### 6.3 Multilinear reformulation: cyclic cohomology (Connes)

Observe that if \( T \in \mathcal{L}^1 \) then
\[ \text{Tr}(\gamma T) = \frac{1}{2} \text{Tr}(\gamma F[F, T]). \]
Indeed
\[ \text{Tr}(\gamma F[F, T]) = \text{Tr}(\gamma (T - FTF)) = \text{Tr}(\gamma T) + \text{Tr}(\gamma T) \]
since \( F\gamma + \gamma F = 0 \).

Both formulas in proposition (6.12) can be obtained from multilinear forms \( \tau \in \text{Hom}(\mathcal{A}^{\otimes n+1}, \mathbb{C}) \).
\[ \tau_{F}(a^{0}, a^{1}, \ldots, a^{n}) = \begin{cases} \text{Tr}(\gamma F[F, a^{0}][F, a^{1}] \ldots [F, a^{n}]) & n \text{ even } > p - 1, \\ \text{Tr}(F[F, a^{0}][F, a^{1}] \ldots [F, a^{n}]) & n \text{ odd } > p - 1. \end{cases} \]

The first comes from (using graded commutators)
\[ \text{Tr}(\gamma F[F, a^{0}][F, a^{1}] \ldots [F, a^{n}]) = \text{Tr}(\gamma F[F, a^{0}][F, a^{1}] \ldots [F, a^{n}]) + \]
\[ + \sum_{i=1}^{n} \text{Tr}(\gamma F a^{0}[F, a^{1}] \ldots [F, a^{i}] \ldots [F, a^{n}]), \]
where the terms in the sum are 0 because
\[ [F, [F, a]] = F[F, a] + [F, a]F = a - FaF + FaF - a = 0. \]
For anti-commutation reasons, the first expression vanishes for \( n \) odd, while the second expression vanishes for \( n \) even.

Element \( \phi \in \text{Hom}(A^\otimes n+1, \mathbb{C}) \) is cyclic if
\[
\phi(a^n, a^0, \ldots, a^{n-1}) = (-1)^n \phi(a^0, a^1, \ldots, a^n)
\]
i.e. \( \lambda_n \phi = \text{Id} \) for cyclic operator \( \lambda_n^{n+1} = \text{Id} \). One has
\[
b_T(a^0, a^1, \ldots, a^{n+1}) = \sum_{i=0}^{n} \tau_F(a^0, \ldots, a^i a^{i+1}, \ldots, a^{n+1}) +
\]
\[+ (-1)^{n+1} \tau_F(a^{n+1}, a^0, a^1, \ldots, a^n) = \sum_{i=1}^{n} (-1)^i \text{Tr}(F[F, a^0] \ldots [F, a^i a^{i+1}] \ldots [F, a^n]) +
\]
\[+ (-1)^{n+1} \text{Tr}(F[F, a^{n+1}, a^0][F, a^1] \ldots [F, a^n]). \]

Now
\[
[F, a^i a^{i+1}] = [F, a^i] a^{i+1} + a^i [F, a^{i+1}].
\]

Because of the alternating signs, terms cancel pairwise if \( n + 1 \) is even
\[
\text{Tr}(F[F, a^0] a^i[F, a^2] \ldots [F, a^{n+1}]) + \text{Tr}(F a^0[F, a^1][F, a^2] \ldots [F, a^{n+1}])
\]
\[- \text{Tr}(F[F, a^0][F, a^1] a^2 \ldots [F, a^{n+1}]) - \text{Tr}(F[F, a^0] a^1[F, a^2] \ldots [F, a^{n+1}]) + \ldots
\]
\[\ldots + (-1)^{n+1} \text{Tr}(F[F, a^{n+1}] a^0[F, a^1] \ldots [F, a^{n+1}]) + (-1)^{n+1} \text{Tr}(F a^{n+1}[F, a^0][F, a^1] \ldots [F, a^{n+1}]). \]

Hence for odd \( n \)
\[
b_T = 0.
\]

For even \( n \)
\[
\text{Tr}(\gamma F[F, a^n][F, a^1] \ldots [F, a^{n-1}]) = \text{Tr}(F[F, a^n][F, a^0] \ldots [F, a^{n-1}]) =
\]
\[- \text{Tr}(F[F, a^0] \ldots [F, a^n]). \]

This leads to the definition of cyclic cohomology, a homology of complex
\[
(C_\lambda(^n\mathfrak{A}), b), \quad C_\lambda^n(\mathfrak{A}) = \text{Hom}_{cont}(\mathfrak{A}^\otimes n+1, \mathbb{C})
\]
for locally convex algebra \( \mathfrak{A} \) (with continuous multiplication).

The fact that \( n \mapsto n + 2 \) leaves formulas in proposition (6.12) unchanged is related to the periodicity operator
\[
S : \text{HC}_\lambda^n(\mathfrak{A}) \mapsto \text{HC}_\lambda^{n+2}(\mathfrak{A})
\]
which in turn is an arrow in Connes long exact sequence
\[
\ldots \mapsto \text{HC}_\lambda^n(\mathfrak{A}) \overset{I}{\mapsto} \text{HH}_\lambda^n(\mathfrak{A}) \overset{B}{\mapsto} \text{HC}_\lambda^{n-1}(\mathfrak{A}) \overset{S}{\mapsto} \text{HC}_\lambda^{n+1}(\mathfrak{A}) \overset{I}{\mapsto} \ldots
\]

For \( \mathfrak{A} = C^\infty(M) \), \( \partial M = 0 \)
\[
\tau(f^0, f^1, \ldots, f^n) = \int_M f^0 df^1 \wedge \ldots \wedge df^n
\]
From Leibniz rule and Stokes theorem
\[ b\tau = 0, \quad \lambda(\tau) = \tau. \]

If \( \omega \in \Omega^{n-k}(M) \) then
\[ \tau_\omega(f^0, \ldots, f^k) := \int_M f^0 df^1 \wedge \ldots \wedge df^k \wedge \omega, \quad d\omega = 0. \]

If \( C \)-\( k \)-current
\[ \tau_c(f^0, \ldots, f^k) = \langle C, f^0 df^1 \wedge \ldots \wedge df^k \rangle, \quad dC = 0. \]

**Theorem 6.13 (Connes).**

\[
\begin{aligned}
\text{HC}^q_{\lambda}(\mathfrak{A}) &\simeq \ker d_q^+ \oplus \text{ker } d_{q-2} \oplus \text{ker } d_{q-4} \oplus \ldots \\
S &
\end{aligned}
\]

where the inclusion \( \ker d_q^+ \hookrightarrow \text{HC}^q_{\lambda}(\mathfrak{A}) \) is
\[ C \mapsto \phi(C(f^0, f^1, \ldots, f^q)) = \langle C, f^0 df^1 \wedge \ldots \wedge df^q \rangle. \]

Compatibility considerations lead to the following normalization for the Connes-Chern character of a K-cycle \( F \) over \( \mathfrak{A} \) of Schatten dimension \( p \).

- For \( n \) odd \( > p - 1 \)
  \[ \tau_n(a^0, a^1, \ldots, a^n) = (-1)^{\frac{n+1}{2}} \left( \frac{n}{2} - 1 \right)^\frac{1}{2} \text{Tr}(F[F,a^0][F,a^1] \ldots [F,a^n]), \]
  \[ S\tau_n = \tau_{n+2} \]

- For \( n \) even \( > p - 1 \)
  \[ \tau_n(a^0, a^1, \ldots, a^n) = \left( \frac{n}{2} \right)^\frac{1}{2} \text{Tr}(\gamma F[F,a^0][F,a^1] \ldots [F,a^n]), \]
  \[ S\tau_n = \tau_{n+2} \]

Homological Chern character is a homomorphism
\[ \text{ch} : K_* (M) \to H^{dR}_* (M; \mathbb{C}) \]

It is a special case of the Connes-Chern character for an algebra
\[ \text{ch}^* K^*(\mathfrak{A}) \to H^{dR}_*(\mathfrak{A}) \]

if one takes \( \mathfrak{A} = C^\infty (M) \). For a cocycle \( (\mathfrak{A}, \mathcal{H}, F) \) representing an element in K-homology one has
\[ \text{ch}^*(\mathfrak{A}, \mathcal{H}, F) := [\phi^n], \]

where \( \phi^n \) is the following cocycle
\[ \phi^n(a^0, a^1, \ldots, a^n) = \text{Tr}(\gamma a^0 [F,a^0] \ldots [F,a^n]) \]

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for \( n \) even.

\[ S[\phi^n] = [\phi^{n+2}] \]

For a Dirac operator \( D \) we can take \( F = D|D|^{-1} \) and then

\[ \text{ch}\_\*(D) = \hat{A}(M) = (\det)^{\frac{1}{2}} \left( \frac{R}{\sinh \frac{R}{2}} \right) \]

If \( \gamma \) is a gradation on \( \mathcal{H} \) i.e.

\[ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \]

then

\[ \text{Index}(D^+) = \text{Tr}(\gamma e^{-tD^2}), \quad t > 0 \]

\[ D^2 = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix} \]

For \( t \to 0^+ \) function \( \text{Tr}(\gamma e^{-tD^2}) \) has an expansion

\[ c_0 + c_1 t + c_2 t^2 + \ldots, \]

where

\[ c_0 = \int_M \omega_\delta(D) \]

and \( \omega_\delta(D) \) is called the local index formula.

### 6.4 Connes cyclic cohomology

\( \text{HC}^*(\mathfrak{A}) \) is defined as the cohomology of a complex \((C_\lambda(\mathfrak{A}), b)\). A cycle representing an element in \( \text{HC}^*(\mathfrak{A}) \) is a triple

\[ (\Omega, d, \int), \]

where \((\Omega, d)\) is a differential graded algebra

\[ \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n, \quad d^2 = 0, \quad \text{(finite length)}, \]

and \( \int \) is a closed graded trace \( \int \Omega^n \to \mathbb{C} \) i.e.

\[ \int \omega_1 \omega_2 = (-1)^{|\omega_1||\omega_2|} \int \omega_2 \omega_1 \text{ (graded trace)}, \]

\[ \int d\omega = 0 \text{ (closed)}. \]

Using homomorphism \( \rho: \mathfrak{A} \to \Omega^0 \) we can write a character of \((\Omega, d, \int)\)

\[ \tau(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \ldots da^n. \]

It is a cyclic cocycle.
Define a **chain** as a triple \((\Omega, \partial \Omega, \int)\), where \(\partial \Omega \subset \Omega\), \(\dim \Omega = n\), \(\dim \partial \Omega = n - 1\), and \(d\) preserves \(\partial \Omega\). There is given a surjective homomorphism \(r: \Omega \to \partial \Omega\) of degree 0 (restriction to the boundary) and
\[
\int d\omega = 0, \ \forall \ \omega \ \text{such that} \ r(\omega) = 0.
\]
A **boundary** of such chain is a cycle \((\partial \Omega, d, \int')\), where for \(\omega' \in \partial \Omega^{n-1}\)
\[
\int' \omega' := \int d\omega, \ \text{for} \ r(\omega) = \omega'.
\]

Two cycles \(\Omega_1, \Omega_2\) are **cobordant**, \(\Omega_1 \sim \Omega_2\) if and only if there exists a chain \((\Omega, \partial \Omega, \int)\) such that
\[
\partial \Omega = \Omega_1 \oplus \tilde{\Omega}_2
\]
where \((\tilde{\Omega}_2, d, \tilde{\int})\) is a cycle in which \(\tilde{\int} \omega = -\int \omega\).

**Theorem 6.14.**

\(\Omega_1 \sim \Omega_2 \iff \tau_2 - \tau_1 = B_0\phi \in \text{im} B_0\)

where the operator \(B_0\) is defined as follows.

\[B_0\phi(a^0, a^1, \ldots, a^n) = \phi(1, a^0, \ldots, a^n) - (-1)^{n+1}\phi(a^0, \ldots, a^n, 1)\]

The operator \(B\) is then equal to \(AB_0\), where \(A\) is the cyclic antisymmetrization

\[(A\phi)(a^0, a^1, \ldots, a^n) := \sum_{i=0}^{n} (-1)^{ni}\phi(a^i, a^{i+1}, \ldots, a^{i-1})\]

The Connes exact sequence

\[\ldots \xrightarrow{B} \text{HC}_{\lambda}^{n-2}(\mathfrak{A}) \xrightarrow{S} \text{HC}_{\lambda}^{n}(\mathfrak{A}) \xrightarrow{L} \text{H}^n(\mathfrak{A}) \xrightarrow{B} \text{HC}_{\lambda}^{n-1}(\mathfrak{A}) \xrightarrow{S} \ldots\]

starts with \(\text{HC}_{\lambda}^{0}(\mathfrak{A}) = \text{H}^0(\mathfrak{A})\). Thus if there is an algebra homomorphism \(\mathfrak{A} \to \mathfrak{A}'\) which induces isomorphism on Hochshild cohomology, then it also induces isomorphism on cyclic cohomology.

We can form a bicomplex \((C^{n,m}, b, B)\) with \(b^2 = 0, B^2 = 0, bB + Bb = 0\), and \(C^{n,m}(\mathfrak{A}) = \mathfrak{A}^{\otimes n-m+1}\). The homology of the total complex is then cyclic cohomology.

**6.5 An alternate route, via the Families Index Theorem**

Set up: \((\mathfrak{A}, \mathcal{H}, D)\), \(D = D^*\) unbounded with

\[\left[D, \mathfrak{A}\right] \subset \mathcal{L}(\mathcal{H}), \ (1 + D^2) \in \mathcal{L}^p\]

In fact we shall assume that \(D\) is invertible with \(D^{-1} \in \mathcal{L}^p\). The bounded version of this K-cycle is given by \((\mathfrak{A}, \mathcal{H}, F)\), where \(F = D|D|^{-1}\) is a phase.

On \(\mathfrak{A}\) one has a norm

\[\|\|a\|\| := \|a\| + \|\left[D, a\right]\|, \ \text{for} \ a \in \mathfrak{A}.
\]

Let \(\mathcal{V} = \mathcal{V}(\mathfrak{A})\) be the span of "vector potentials", that is

\[\mathcal{V} := \left\{ A = \sum_i a_i[D, b_i] \mid a_i, b_i \in \mathfrak{A}, \ A = A^* \right\}.
\]
Let $\mathcal{U} = \mathcal{U}(\mathfrak{A})$ be the gauge group, that is

$$\mathcal{U} = \mathcal{U}(\mathfrak{A}) := \{ u \in \text{GL}_1(\mathfrak{A}) \mid u^*u = uu^* = 1 \},$$

acting on $\mathcal{V}$ by (affine action)

$$u \cdot A := u[D, u^*] + uAu^* = u(D + A)u^* - D.$$ 

Denoting $D_A := D + A$ one has

$$D_{u \cdot A} = uD_Au^*.$$

**Fact 6.15.** $D_A$ has the same dimension as $D$ and $D_A^* = D_A$. Also $\ker D_A = \ker(\text{Id} + D^{-1}A)$, hence is finite dimensional.

Let

$$\mathcal{V}_{inj} := \{ A \in \mathcal{V} \mid D_A \text{ injective} \} \subset \mathcal{V}$$

It is an open subset with respect to $\|\cdot\|$. For $A \in \mathcal{V}_{inj}$ operator $D_A$ is invertible with

$$D_A^{-1} = (1 + D^{-1}A)^{-1}D^{-1} \in \mathcal{L}^p.$$

Graded trivial vector bundle over $\mathcal{V}_{inj}$

$$\tilde{\mathcal{H}}^\pm := \mathcal{V}_{inj} \times \mathcal{H}^\pm.$$

Superconnection is an operator $d + \tilde{D}$, where

$$\tilde{D}: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}, \text{ is in the fiber } \tilde{D}_A = D_A: \mathcal{H}^\pm \to \mathcal{H}^\pm.$$

Curvature

$$\mathcal{R} := (\gamma d + \tilde{D})^2 = \gamma d\tilde{D} + \tilde{D}d + \tilde{D}^2 = [\gamma d, \tilde{D}] + \tilde{D}^2.$$ 

Explicit expression of $\tilde{D}' = [d, \tilde{D}] \in \Omega^1(\mathcal{V}_{inj}, \tilde{\mathcal{H}})$:

$$d: \Omega^p(\mathcal{V}_{inj}, \tilde{\mathcal{H}}) \to \Omega^{p+1}(\mathcal{V}_{inj}, \tilde{\mathcal{H}})$$

$$(d\omega)(\tilde{X}_0, \ldots, \tilde{X}_{p+1}) = \sum_{i=0}^{p} \tilde{X}_i\omega(\tilde{X}_0, \ldots, \hat{\tilde{X}}_i, \ldots, \tilde{X}_{p+1})$$

(commutators vanish), where

$$\tilde{X}_A f := \frac{d}{dt}|_{t=0} f(A + tX), \ X \in \mathcal{V}.$$ 

One has with $F: \mathcal{V}_{inj} \to \mathcal{L}(\mathcal{H}), F(A) := D + A$

$$\gamma d(\tilde{D}\omega) = \gamma dF \wedge \omega,$$

Hence

$$\tilde{D}'(\omega) = dF \wedge \omega, \ dF_A(\tilde{X}_A) = X,$$

$$\tilde{D}'(\omega)_A(X_0, \ldots, X_{p+1}) = \sum_{i=0}^{r} (-1)^i \underbrace{X_i}_{\in \mathcal{L}(\mathcal{H})} \underbrace{\omega_A(X_0, \ldots, \hat{X}_i, \ldots, X_{p+1})}_{\in \mathcal{H}}$$

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(Super) Chern form

\[ \Omega_t^{(n)} := \text{Tr} \left( \gamma e^{-(tD' + t^2\bar{D}^2)} \right)^{(n)} = \text{Tr} \left( \gamma e^{-t\bar{R}^2} \right)^{(n)} = \]

\[= (-t)^n \int_{\Delta_n} \text{Tr} \left( e^{-s_1t^2\bar{D}^2} D'e^{-s_2t^2\bar{D}^2} \ldots e^{-s_n(s_{n-1})t^2\bar{D}^2} D'e^{-(1-s_n)t^2\bar{D}^2} \right) ds_1 ds_2 \ldots ds_n, \]

and the integration is over a simplex

\[\Delta_n := \{0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq 1 \mid s_1 + s_2 + \ldots + s_n = 1\}\]

One has

\[ \frac{d}{ds} (e^{s(A+B)} e^{-sB}) = e^{s(A+B)} A e^{-sB} \]

\[ e^{u(A+B)} = e^{uB} + \int_0^u e^{s(A+B)} A e^{(u-s)B} ds. \]

[TO BE CONTINUED ...]

### 6.6 Index theory for foliations

Let \((M^m, \mathcal{F})\) be a foliated manifold. To define an index in noncommutative geometry we have to complete definitions of the following tasks

1. transverse coordinates,
2. analog of elliptic operator,
3. index pairing between K-theory and K-homology.

Foliation can be described using 1-cocycle \((V_i, f_i, g_{ij})\), where

\[ f_i: V_i \to U_i \subset \mathbb{R}^n, \ n = \text{codim} \mathcal{F} \] are surjective submersions,

and \(g_{ij}: f_j(V_i \cap V_j) \to f_i(V_i \cap V_j)\) are diffeomorphisms such that

\[ g_{ij} \circ g_{jk} = g_{ik}. \]

Above cocycle gives a grupoid \(\Gamma = \{g_{ij}\}\) which leads to the algebra of foliation

\[ \mathfrak{A}_\Gamma := C^\infty_c(FM) \rtimes \Gamma \]

\[ f u_\phi \cdot g u_\psi = f g \phi^{-1} u_{\phi \psi}, \ \phi, \psi \in \Gamma. \]

where \(FM = J^1(M)\) is a frame bundle. This gives a transverse coordinates. The advantage in working with frame bundle is that \(FM\) has a natural volume form. It is paralelizable (i.e. \(TFM\) is trivial). One has a principal bundle

\[ \text{GL}_n(\mathbb{R}) \to FM \]

\[ \pi \]

\[ M \]

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One has vertical vector fields $Y^j_i$ coming from the $\text{GL}_n(\mathbb{R})$ action, and when chooses a connection, also horizontal vector fields $X_k$. Let $\{\theta^k, \omega^j_i\}$ be the dual basis of differential forms. Then

$$\Lambda \omega^j_i \wedge \Lambda \theta^k$$

is an invariant volume form.

For our second task we have to give up ellipticity. Consider a quotient bundle

$$ FM/ \text{SO}(n) =: PM $$

The fiber $PM_x$ is the space of all Euclidean structures on $T_xM$

$$ \langle \zeta, \eta \rangle = \langle a\zeta, a\eta \rangle, \ a \in \text{SO}(n). $$

Section of $PM$ are all Riemannian metrics on $TM$. Let

$$ \mathcal{V} \subset TP M = \ker \pi_* $$

be the vertical subbundle (vectors tangent to the fibers). On the quotient $\text{GL}_n(\mathbb{R})/\text{SO}(n)$ there is a metric, and determines a metric on $\mathcal{V}$.

$$ TP M/\mathcal{V} =: \mathcal{N} $$

The horizontal bundle $\mathcal{N}$ has a tautological Riemannian structure. Indeed, $p \in PM$ is an Euclidean structure for $T_{\pi(p)}M$, and $\mathcal{N}_p$ is identified with $T_{\pi(p)}M$ by $\pi_*$. The bundle $TP M$ has a decomposition into vertical and horizontal part, $TP M = \mathcal{V} \oplus \mathcal{N}$. The Hilbert space

$$ L^2(\Lambda T^* PM, \text{vol}_P) $$

where $\text{vol}_P$ is a volume form induced by canonical volume form on $FM$, decomposes also as a tensor product of corresponding Hilbert spaces

$$ L^2(\Lambda T^* PM) = L^2(\Lambda \mathcal{V}^*) \otimes L^2(\Lambda \mathcal{N}^*). $$

On this two parts we have operators

- On $L^2(\Lambda \mathcal{V}^*)$ with vertical differential $d_V$

  $$ Q_V := i(d_V + d_V^*)(d_V - d_V^*) = -i(d_V d_V^* + d_V^* d_V) $$

- On $L^2(\Lambda \mathcal{N}^*)$ with horizontal differential $d_H$

  $$ Q_H := d_H + d_H^* $$

On the whole $L^2(\Lambda T^* PM)$ we put $Q = Q_V \oplus \gamma_V Q_H$, where $\gamma_V$ is the grading of the vertical signature. Operator $Q = Q^*$ is called **hypoelliptic signature operator**. We have a spectral triple ($\mathfrak{A}_\Gamma, \mathcal{H}, D$), where $D$ is determined by the equation $Q = D|D|$. 55
For \( a \in \mathfrak{A} \] \([D, a] \in \mathcal{L}(\mathcal{H}) \) and \((1 + D^2)^{-\frac{1}{2}} \in \mathcal{L}^p(\mathcal{H}) \) for \( p = \dim V + 2n \), where \( \dim M = n \). The K-cycle \((\mathfrak{A}, \mathcal{H}, D)\) gives an element in \( K^*_\text{Diff}_M(\mathfrak{A}) \) (\( \text{Diff}_M \)-equivariant K-cycle). Its character \( \text{ch}_s(D) \in HC^*_\mathfrak{A}(\mathfrak{A}_\Gamma) \) can be expressed in terms of residues of spectrally defined zeta-functions, and is given by a cocycle \( \{\phi_n\} \) in the \((b, B)\)-bicomplex of \( \mathfrak{A}_\Gamma \) whose components are of the following form

\[
\text{Res}_{s=0} \text{Tr}(a^0[a^1, D]^{(k_1)} \ldots [a^n, D]^{(k_n)}|D|^{-n-2|k|-s})
\]

which we denote by

\[
\int \text{Tr}(a^0[a^1, D]^{(k_1)} \ldots [a^n, D]^{(k_n)}|D|^{-n-2|k|-s})
\]

\[
\phi_n(a^0, \ldots, a^n) = \sum_k c_{n,k} \int a^0[Q, a^1]^{(k_1)} \ldots [Q, a^n]^{(k_n)}|Q|^{-n-2|k|}
\]
Chapter 7

Hopf cyclic cohomology

7.1 Preliminaries

Lecture given by Piotr Hajac

7.1.1 Cyclic cohomology in abelian category

Our task is to understand cup product for Hopf-cyclic cohomology with coefficients, that is mapping

\[ HC_H^m(C; M) \otimes HC_H^n(A; M) \rightarrow HC_{m+n}(A; M). \]

Consider a category \( C \), with finite sets \([n] := \{0, 1, \ldots, n\}\) for \( n \in \mathbb{N} \) as objects, and morphism which preserve order. To describe a cyclic structure we introduce following morphisms

- **Face**
  \[ [n-1] \xrightarrow{i} [n], \ 0 \leq i \leq n, \]
  - injection which misses \( i \).

- **Degeneracy**
  \[ [n+1] \xrightarrow{j} [n], \ 0 \leq j \leq n, \]
  - surjection which sends both \( j \) and \( j+1 \) to \( j \).

- **Cyclic operator**
  \[ [n] \xrightarrow{\tau_n} [n] \]
  - cyclic shift to the right.

The morphism above satisfy following identities, which we can group to obtain successive complications of our category.

- **Presimplicial category.**

  \[ \text{Mor}(C) := \{ \delta_i^{(n)} \mid 0 \leq i \leq n, \ n \in \mathbb{N} \}, \]

  with

  \[ \delta_j \delta_i = \delta_i \delta_j, \ j > i. \]
• Simplicial category.

\[
\text{Mor} (\mathcal{C}) := \{ \delta_i^{(n)} , \sigma_j^{(m)} \mid 0 \leq i \leq n, \ 0 \leq j \leq m, \ n, m \in \mathbb{N} \},
\]

with additional identities

\[
\sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \ i \leq j,
\]

\[
\sigma_j \delta_i = \begin{cases} 
\delta_i \sigma_{j-1}, & i < j, \\
\text{id}_{[n]}, & i \in \{j, j+1\}, \\
\delta_{i-1} \sigma_j, & i > j + 1
\end{cases}
\]

• Precyclic category.

\[
\text{Mor} (\mathcal{C}) := \{ \delta_i^{(m)} , \tau_n \mid 0 \leq i \leq m, \ m, n \in \mathbb{N} \},
\]

with the identities as for presimlicial category and

\[
\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \ 1 \leq i \leq n.
\]

• Cyclic Category.

\[
\text{Mor} (\mathcal{C}) := \{ \delta_i^{(m)} , \sigma_j^{(l)} , \tau_n \mid 0 \leq i \leq m, \ 0 \leq j \leq l, \ m, l, n \in \mathbb{N} \},
\]

with all above identeties and

\[
\tau_n \sigma_0 = \sigma_n \tau_{n+1},
\]

\[
\tau_n \sigma_j = \sigma_{j-1} \tau_{n+1}, \ 1 \leq j \leq n.
\]

Now, let \( \mathcal{A} \) be an abelian category, and \( F : \mathcal{C} \to \mathcal{A} \) a functor. It means that we have a sequence of objects, and morphisms

\[
A_n \xrightarrow{\delta_i} A_{n-1} \xrightarrow{\tau_n} A_n \xrightarrow{\sigma_i} A_{n+1}.
\]

Define

\[
b_n := \sum_{i=0}^n (-1)^i \delta_i, \quad b'_n := \sum_{i=0}^{n-1} (-1)^i \delta_i,
\]

\[
\lambda_n := (-1)^n \tau_n, \ n \in \mathbb{N}.
\]

These morphisms satisfy the following identities

\[
b_{n+1} b_n = 0, \ (1 - \lambda_n) b_n = b'_n (1 - \lambda_{n-1}).
\]

Consider a diagram

\[
\begin{array}{c}
\text{ker}_{n+1} \longrightarrow A_{n+1} \xrightarrow{1 - \lambda_{n+1}} A_{n+1} \\
\downarrow b_{n+1} \quad \downarrow b_{n+1} \quad \downarrow b'_{n+1} \\
\text{ker}_n \longrightarrow A_n \xrightarrow{1 - \lambda_n} A_n \\
\downarrow b_n \quad \downarrow b_n \quad \downarrow b'_n \\
\text{ker}_{n-1} \longrightarrow A_{n-1} \xrightarrow{1 - \lambda_{n-1}} A_{n-1}
\end{array}
\]
The composition $b_{n+1}b_n = 0$, so we have a complex

```
ker_{n-1} \xrightarrow{b_n} ker_n \xrightarrow{b_{n+1}} ker_{n+1}
```

Define the cyclic cohomology of the complex $(A_\bullet, b_n)$ as the cokernel of the unique map $\phi_n$

$$HC^n(F) := HC^n(A_\bullet) := \text{coker } \phi_n.$$  

Define another operator

$$N_n := \sum_{i=0}^{n} (\lambda_n)^i, \quad n \in \mathbb{N}.$$  

Now one can form a bicomplex

```
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\overset{b_3}{A_2} & \overset{-b'_3}{A_2} & \overset{b_3}{A_2} & \overset{-b'_3}{A_2} & \overset{b_3}{A_2} & \cdots \\
\underset{1-\lambda_2}{A_2} & \underset{1-\lambda_2}{A_2} & \underset{1-\lambda_2}{A_2} & \underset{1-\lambda_2}{A_2} & \cdots \\
\overset{b_2}{A_1} & \overset{-b'_2}{A_1} & \overset{b_2}{A_1} & \overset{-b'_2}{A_1} & \overset{b_2}{A_1} & \cdots \\
\underset{1-\lambda_1}{A_1} & \underset{1-\lambda_1}{A_1} & \underset{1-\lambda_1}{A_1} & \underset{1-\lambda_1}{A_1} & \cdots \\
\overset{b_1}{A_0} & \overset{-b'_1}{A_0} & \overset{b_1}{A_0} & \overset{-b'_1}{A_0} & \overset{b_1}{A_0} & \cdots \\
\underset{1-\lambda_0}{A_0} & \underset{1-\lambda_0}{A_0} & \underset{1-\lambda_0}{A_0} & \underset{1-\lambda_0}{A_0} & \cdots \\
\end{array}
```

Then the cohomology of the total complex is the cyclic cohomology of the functor $F: C \to A$

$$HC^n(F) = H^n(\text{Tot } A_\bullet).$$

### 7.1.2 Hopf algebras

Summary of notations.

- Coalgebra $(C, \Delta, \epsilon)$

```
\begin{array}{cccc}
C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes C \otimes C \\
\downarrow{\Delta} & & \downarrow{\text{id} \otimes \Delta} & & \downarrow{\Delta \otimes \text{id}} \\
C \otimes C & \xrightarrow{\text{id} \otimes \Delta} & C \otimes C \otimes C \\
\end{array}
```

```
\begin{array}{cccc}
C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{\epsilon \otimes \text{id}} & C \\
\downarrow{\Delta} & & \downarrow{\text{id}} & & \downarrow{\epsilon} \\
C \otimes C & \xrightarrow{\text{id} \otimes \epsilon} & C \\
\end{array}
```
• Comodule \((M, \Delta_R)\)

\[
\begin{array}{ccc}
M & \xrightarrow{\Delta_R} & M \otimes C \\
\downarrow{\Delta_R} & & \downarrow{\Delta_R \otimes \text{id}} \\
M \otimes C & \xrightarrow{\text{id} \otimes \Delta_R} & M \otimes C \otimes C
\end{array}
\]

\[
\begin{array}{ccc}
M & \xrightarrow{\Delta_R} & M \otimes C \\
\downarrow{\text{id}} & & \downarrow{\epsilon \otimes \text{id}} \\
M & & M
\end{array}
\]

• Bicomodule \((M, \Delta_L, \Delta_R)\)

\[
\begin{array}{ccc}
M & \xrightarrow{\Delta_R} & M \otimes C \\
\downarrow{\Delta_L} & & \downarrow{\Delta_L \otimes \text{id}} \\
M \otimes C & \xrightarrow{\text{id} \otimes \Delta_R} & C \otimes M \otimes C
\end{array}
\]

• Hopf algebra \((H, m, 1, \Delta, \epsilon, S)\), where
  
  – \((H, m, 1)\) algebra,
  – \((H, \Delta, \epsilon)\) coalgebra,
  – \(\Delta, \epsilon\) are algebra homomorphisms,
  – Convolution product \(f \ast g\)
    \[
    f \ast g: H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{m} H,
    \]
  – Antipode \(S\)
    \[
    S \ast \text{id} = 1 \epsilon = \text{id} \ast S.
    \]

Properties of \(S\):
  
  • if exists, it is unique,
  • it is an antialgebra map: \(S(ab) = S(b)S(a)\),
  • it is an anticoalgebra map: \(\Delta \circ S = (S \otimes S) \circ \Delta^{op}\),
  • if there exists \(S^{-1}\), it has the above properties and satisfies
    \[
    S^{-1} \ast_{\text{cop}} \text{id} = 1 \epsilon = \text{id} \ast_{\text{cop}} S^{-1}.
    \]

Sweedler notation:
\[
\Delta h = \sum_i a_i \otimes b_i =: h^{(1)} \otimes h^{(2)}.
\]

If we treat multiple tensor products as trees, then we can forget how the tree was constructed.
\[
\Delta^2 h = h^{(1)(1)} \otimes h^{(1)(2)} \otimes h^{(2)} = h^{(1)} \otimes h^{(2)(1)} \otimes h^{(2)(2)} = h^{(1)} \otimes h^{(2)} \otimes h^{(3)}.
\]
\[
\Delta_R m = m^{(0)} \otimes m^{(1)}, \quad \Delta_L m = m^{(-1)} \otimes m^{(0)}.
\]
7.1.3 Motivation for Hopf-cyclic cohomology

If $D$ is a Dirac operator, $E$ idempotent, then there exists an index pairing

$$(\text{ch}^*(D), \text{ch}_v(E)) =: \text{Index}(DE).$$

For the transverse geometry of a codim $= n$ foliation

$$\text{ch}^*(D)(a_0, \ldots, a_m) = \text{tr}_\delta(a_0 h_1(a_1) \ldots h_m(a_m)),$$

where $h_i \in \mathcal{H}_n$ - the universal Hopf algebra for codim $= n$ foliations, $\delta: H \to k$- character, $\text{tr}_\delta - \delta$-invariant trace.

$$\mathcal{H}_n \otimes A \to A$$

$$h(ab) = h^{(1)}(a)h^{(2)}(b), \quad 1_H(a) = a.$$

In particular

$$\Delta(g) = g \otimes g \text{ (group-like element)} \implies g(ab) = g(a)g(b),$$

$$\Delta x = x \otimes 1 + 1 \otimes x \text{ (primitive element)} \implies x(ab) = x(a)b + ax(b).$$

One has

$$\text{tr}_\delta(a_0 h_1(a_1) \ldots h_m(a_m)) = (-1)^m \text{tr}_\delta(a_0 h_1(a_0) \ldots h_m(a_{m-1}))$$

$$= (-1)^m \text{tr}_\delta(h_1(a_0) \ldots h_m(a_{m-1})a_m).$$

In particular

$$\text{tr}_\delta(h(a)) = \delta(h) \text{tr}_\delta(a),$$

$$\text{tr}_\delta(h(a)b) = \text{tr}_\delta(h^{(1)}(a)(h^{(2)}S(h^{(3)}))(b)) = \text{tr}_\delta(h^{(1)}(a)h^{(2)}(S(h^{(3)}))(b)) =$$

$$= \text{tr}_\delta(h^{(1)}(aS(h^{(2)}))(b)) = \delta(h^{(1)}) \text{tr}_\delta(aS(h^{(2)}))(b) =$$

$$= \text{tr}_\delta(a(\delta \ast S)(h))(b).$$

Hence

$$\text{tr}_\delta(a_0 h_1(a_1) \ldots h_m(a_m)) = (-1)^m \text{tr}_\delta(a_0(\delta \ast S)(h_1)(h_2(a_1) \ldots h_m(a_{m-1})a_m))$$

Denote

$$h_1 \otimes \ldots \otimes h_m = (-1)^m (\delta \ast S)(h_1)(h_2 \otimes \ldots h_m \otimes 1) =: (-1)^m \tau_m(h_1 \otimes \ldots \otimes h_m).$$

For an element $\sigma \in \mathcal{H}_n$ such that $\Delta \sigma = \sigma \otimes \sigma$, $\delta(\sigma) = 1$

$$\text{tr}^2_\delta(ab) = \text{tr}^2_\delta(b\sigma(a))$$

which implies

$$\tau_m(h_1 \otimes \ldots \otimes h_m) = (\delta \ast S)(h_1)(h_2 \otimes \ldots \otimes h_m \otimes \sigma).$$

$$(-1)^m \text{tr}_\delta(h_1(a_0)h_2(a_1) \ldots h_m(a_{m-1})a_m) = (-1)^m \text{tr}_\delta(a_0(\delta \ast S)(h_1)(h_2(a_1) \ldots h_m(a_{m-1})a_m)) =$$

$$= (-1)^m \text{tr}_\delta(a_0 h(b)).$$

$$(-1)^m (\delta \ast S)(h_1)(h_2 \otimes \ldots \otimes h_m \otimes 1) = \lambda_m(h_1 \otimes \ldots \otimes h_m).$$
Now one has to check that $\tau_m^{m+1} = \text{id}$. For $m = 1$

$$\tau_1^2(h) = \tau_1((\delta * S)(h)\sigma) = \delta(h^{(1)})(\delta * S)(S(h^{(2)})\sigma)\sigma =$$

$$\delta(h^{(1)})\delta(S(h^{(3)}))\sigma^{-1}S^2(h^{(2)})\sigma = \sigma^{-1}(\delta * S^2 * \delta^{-1})(h)\sigma = h$$

Denote $S_\sigma^\delta(h) := (\delta * S)(h)\sigma$.

Now from $(\tau_1)^2 = (S_\sigma^\delta)^2 = \text{id}$ one can deduce after computation that for all $m \quad \tau_m^{m+1} = \text{id}$ (Connes-Moscovici). This yields a new cyclic complex

$$(H^\otimes m, \delta, \sigma, \tau_m)_{m \in \mathbb{N}}$$

for any Hopf algebra $H$ equipped with modular pair in involution (MPII) $(\delta, \sigma)$. For example, if $S^2 = \text{id}$, then $(\epsilon, 1)$ is a modular pair in involution.

Example 7.1. Let $H = \mathcal{H}_1$ be an universal algebra for codim = 1 foliations. First take a Lie algebra $\mathfrak{h}_1$ with generators $X, Y, \lambda_n, n \in \mathbb{N}$ satisfying

$$[Y, X] = X,$$

$$[X, \lambda_n] = \lambda_{n+1},$$

$$[Y, \lambda_n] = n\lambda_n,$$

$$[\lambda_n, \lambda_m] = 0 \quad \forall \quad n, m \geq 1.$$  

Then form an universal enveloping algebra $\mathcal{H}_1 := U(\mathfrak{h}_1)$. The coproduct on $\mathcal{H}_1$ id uniquely determined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \lambda_1 \otimes Y,$$

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y,$$

$$\Delta(\lambda_1) = \lambda_1 \otimes 1 + 1 \otimes \lambda_1.$$

The counit

$$\epsilon(X) = \epsilon(Y) = \epsilon(\lambda_1) = 0.$$

The antipode

$$S(Y) = -Y, \quad S(\lambda_1) = -\lambda_1,$$

$$S(X) = -X + \lambda_1 Y.$$

Now take $\sigma = 1$,

$$\delta(X) = 0, \quad \delta(\lambda_1) = 0, \quad \delta(Y) = -1.$$

One has to check that

$$\delta(h^{(1)})S^2(h^{(2)})\delta(S(h^{(3)})) = h.$$

On generators

$$Y^{(1)} \otimes Y^{(2)} \otimes Y^{(3)} = Y \otimes 1 \otimes 1 + 1 \otimes Y \otimes 1 + 1 \otimes 1 \otimes Y,$$

$$\delta(Y) + S^2(Y) - \delta(Y) = Y.$$

Similarly for $\lambda_1$.

$$X^{(1)} \otimes X^{(2)} \otimes X^{(3)} =$$

$$= X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X + 1 \otimes \lambda_1 \otimes Y + \lambda_1 \otimes Y \otimes 1 + \lambda_1 \otimes 1 \otimes Y,$$

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\( S^2(X) + \delta(S(X)) - S^2(\lambda_1)\delta(Y) = S(-X + \lambda_1 Y) + \lambda_1 = 0 \)

\( = X - \lambda_1 Y + S(Y)S(\lambda_1) + \lambda_1 = \)

\( \lambda_1 Y = X + \lambda_1 - \lambda_1 = X. \)

Thus \((\delta, 1)\) is a modular pair in involution.

### 7.1.4 Hopf-cyclic cohomology with coefficients

**Motivation:**
- Short proof of \( \tau_2^1 = \text{id} = \tau_{n+1}^n = \text{id} \).
- Constructive common denominator for all known cyclic theories.
- Non-trivial coefficients are geometrically desired and occur in "real life" in the number theory work of Connes-Moscovici.

**Simplicial structure in coalgebra case:**

\[
C^n(C, M) := M \otimes C \otimes C^\otimes n, \quad n \in \mathbb{N},
\]

\( C \) is an \( H \)-module coalgebra

\[
\Delta(hc) = h^{(1)}c^{(1)} \otimes h^{(2)}c^{(2)}, \quad \epsilon(hc) = \epsilon(h)\epsilon(c).
\]

\( M \) is a \( C \)-bimodule

\[
\Delta_R(m \otimes c) = (m \otimes c^{(1)}) \otimes c^{(2)},
\]

\[
\Delta_L(m \otimes c) = m^{(-1)}c^{(1)} \otimes (m^{(0)} \otimes c^{(2)}).
\]

The standard example yields

\[
\delta_i(m \otimes c_0 \otimes \ldots \otimes c_{n-1}) = m \otimes c_0 \ldots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes \ldots \otimes c_{n-1},
\]

\[
\delta_n(m \otimes c_0 \otimes \ldots \otimes c_{n-1}) = m^{(0)} \otimes c_0^{(2)} \otimes c_1 \otimes \ldots \otimes c_{n-1} \otimes m^{(-1)}c_0^{(1)},
\]

\[
\sigma_i(m \otimes c_0 \otimes \ldots \otimes c_{n+1}) = m \otimes c_0 \otimes \ldots \otimes \epsilon(c_{i+1}) \otimes \ldots \otimes c_{n+1}.
\]

**Simplicial structure in algebra case:**

\[
C^n(A, M) := \text{Hom}(M \otimes A \otimes A^\otimes n, k), \quad n \in \mathbb{N}.
\]

\( A \) is an \( H \)-module algebra

\[
h(ab) = (h^{(1)}a)(h^{(2)}b), \quad h1 = \epsilon(h).
\]

\( M \) is a left \( H \)-comodule

\[
\text{Hom}(M \otimes A \otimes A^\otimes n, k) \simeq \text{Hom}(A^\otimes n, \text{Hom}(M \otimes A, k)).
\]

\( M \otimes A \) is an \( A \)-bimodule

\[
(m \otimes a)b = m \otimes ab, \quad b(m \otimes a) = m^{(0)} \otimes (S^{-1}(m^{(-1)}b)a
\]

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The standard example yields
\[(\delta_i f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m \otimes a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n),\]
\[(\delta_n f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m^{(0)}(S^{-1}(m^{(-1)})a_0) \otimes \ldots \otimes a_{n-1}),\]
\[(\sigma_i f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m \otimes a_0 \otimes \ldots \otimes a_i 1 \otimes a_{i+1} \otimes \ldots \otimes a_n).\]

Paracyclic structures:
For \(\{C^n(A, M)\}_{n \in \mathbb{N}}\)
\[(\tau_n f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m^{(0)}(S^{-1}(m^{(-1)})a_0) \otimes a_0 \otimes \ldots \otimes a_{n-1}).\]
For \(\{C^n(C, M)\}_{n \in \mathbb{N}}\)
\[\tau_n(m \otimes c_0 \otimes \ldots \otimes c_n) = m^{(0)} \otimes c_1 \otimes \ldots \otimes c_n \otimes m^{(-1)}c_0.\]

Invariant complexes:
\[C^n_H(A, M) := \text{Hom}_H(M \otimes A^{\otimes n+1}, k),\]
\[M \in \mathcal{H} \mathcal{M}_H, \quad (m \otimes \bar{a}) h = mh^{(1)} \otimes S(h^{(2)})\bar{a}, \quad k = k_\ell\]
\[C^n_H(C, M) := M \otimes_H C^{\otimes n+1},\]
\[M \in \mathcal{H} \mathcal{M}_H, \quad h(c_0 \otimes \ldots c_n) = h^{(1)}c_0 \otimes \ldots \otimes h^{(n+1)}c_n.\]

Cyclic structures:
We say that a bimodule \(M \in \mathcal{H} \mathcal{M}_H\) is \textbf{stable} iff.
\[\forall m \in M \quad m^{(0)}m^{(-1)} = m.\]

It is \textbf{anti-Yetter-Drinfeld} iff.
\[\Delta_L(mh) = S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)}h^{(2)}, \quad \forall m, h.\]

\textbf{Theorem 7.2}. If \(M\) is a stable anti-Yetter-Drinfeld module (SAYD), then the formulas for \(\delta_i, \sigma_i\) and \(\tau_n\) define cyclic structures on \(C^n_H(A, M)\) and \(C^n_H(C, M)\).

Shortly
- anti-Yetter-Drinfeld \(\implies\) \(\tau_n\) is well defined,
- stability \(\implies\) \(\tau_n^{n+1} = \text{id}\).

\textbf{Proof}. First we check that \(\tau_n\) is well defined, that is
\[\tau_n(mh \otimes c_0 \otimes \ldots \otimes c_n) = \tau_n(m \otimes h(c_0 \otimes \ldots \otimes c_n)),\]
\[(mh)^{(0)} \otimes_H (c_1 \otimes \ldots \otimes c_n \otimes (mh)^{-1}c_0) = m^{(0)} \otimes_H (h^{(2)}(c_1 \otimes \ldots \otimes c_n) \otimes m^{(-1)}h^{(1)}c_0),\]
hence it suffices to prove the following identity
\[(mh)^{(0)} \otimes_H (1 \otimes (mh)^{-1}) = m^{(0)} \otimes_H (h^{(2)} \otimes m^{(-1)}h^{(1)}).\]

Take
\[M \otimes_H (H. \otimes H.) \quad \text{(diagonal structure)}\]
and morphism

\[ H \otimes H. \xrightarrow{\Phi} H. \otimes H \text{ (multiplication on the first term)} \]

\[ \Phi(h \otimes k) = h^{(1)} \otimes S(h^{(2)})k, \]

\[ \Phi^{-1}(h \otimes k) = h^{(1)} \otimes h^{(2)}k. \]

Now

\[ \Phi^{-1}(l(h \otimes k)) = \Phi^{-1}(lh \otimes k) = l \Phi^{-1}(h \otimes k). \]

Consider

\[ M \otimes H (H \otimes H) \xrightarrow{\text{id} \otimes_H \Phi} M \otimes H (H \otimes H) \simeq M \otimes H. \]

\[ (mh)^{(0)} \otimes (mh)^{(-1)} = m^{(0)}h^{(2)} \otimes S(h^{(3)})m^{(-1)}h^{(1)}. \]

-anti-Yetter-Drinfeld condition.

\[ \tau_n^{n+1}(m \otimes_H c_0 \otimes \ldots \otimes c_n) = \tau_n^{n}(m^{(0)} \otimes_H c_1 \otimes \ldots \otimes c_n \otimes m^{(-1)}c_0) = m^{(0)} \otimes m^{(-1)}(c_0 \otimes \ldots \otimes c_n) = m^{(0)}m^{(-1)} \otimes c_0 \otimes \ldots \otimes c_n = m \otimes_H c_0 \otimes \ldots \otimes c_n, \]

where in the last equality we used stability of \( M \).

7.1.5 Special cases


\[ C = H, \quad M = \sigma k_\delta \]

Then \( \sigma k_\delta \) is SAYD iff. \( (\delta, \sigma) \) is MPII. Let \( F \) be the isomorphism

\[ F : k \otimes_H (H. \otimes H^{\otimes n}) \xrightarrow{\simeq} H^{\otimes n}. \]

Then for \( \tilde{f} \in H^{\otimes n} \)

\[ \tau_n(h_1 \otimes \ldots \otimes h_n) = (F \circ \tau_n \circ F^{-1})(\tilde{h}) = (F \circ \tau_n)(1 \otimes_H \Phi^{-1}(1 \otimes \tilde{h})) = F(1 \otimes_H (h \otimes \sigma)) = 1 \otimes_H \Phi(h_1 \otimes \ldots \otimes h_n \otimes \sigma) = 1 \otimes_H h_1^{(1)} \otimes S(h_1^{(2)})(h_2 \otimes \ldots \otimes h_n \otimes \sigma) = \delta(h_1^{(1)})S(h_1^{(2)})(h_2 \otimes \ldots \otimes h_n \otimes \sigma). \]

2. \( \text{tr}_\delta^\sigma \in \text{HC}^0_H(A; \sigma k_\delta) \)

3. Characteristic map of Connes-Moscovici

\[ \text{HC}^m_H(H; \sigma k_\delta) \otimes \text{HC}^0_H(A; \sigma k_\delta) \rightarrow \text{HC}^m(A), \]

\[ h_1 \otimes \ldots \otimes h_m \mapsto ((a_0 \otimes \ldots \otimes a_m) \mapsto \text{tr}_\delta^\sigma(a_0 h_1(a_1) \otimes h_m(a_m))) \]

4. The \( n > 0 \) and \( \dim M > 1 \) already applied in Connes-Moscovici work on number theory.

5. \( \text{HC}^m_k(A; k) = \text{HC}^m(A) \)
6. Twisted cyclic cohomology

$$\text{HC}^*_{k[\sigma, \sigma^{-1}]}(A; \kappa).$$

**Lemma 7.3.**

$\sigma k_\delta$ is SAYD $\iff$ $(\delta, \sigma)$ is MPII.

**Proof.**

$$m^{(0)}m^{(-1)} = m \iff 1 \cdot \sigma = \delta(\sigma) = 1,$$

$$(mh)^{(-1)} \otimes (mh)^{(0)} = S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)}h^{(2)}$$

$$\sigma \delta(h) = S(h^{(3)})\sigma h^{(1)} \delta(h^{(2)})$$

$$L(h) = R(h) \iff (L \ast_{\text{op}} S^{-1})(h) = (R \ast_{\text{op}} S^{-1})(h)$$

$$L(h^{(2)})S^{-1}(h^{(1)}) = R(h^{(2)})S^{-1}(h^{(1)})$$

$$S^{-\delta}_\sigma(h) = \sigma \delta(h^{(2)})S^{-1}(h^{(1)}) = S(h^{(2)})\sigma \delta(h^{(1)}) =: S^{\sigma}_\delta(h)$$

By direct computation

$$S^{-\delta}_\sigma \circ S^{\sigma}_\delta = \text{id} = S^{\sigma}_\delta \circ S^{-\delta}_\sigma,$$

i.e.

$$S^{-\delta}_\sigma = (S^{\sigma}_\delta)^{-1}.$$  

Therefore

$$\text{AYD} \iff (S^{\sigma}_\delta)^{-1} = S^{\sigma}_\delta$$

$$(S^{\sigma}_\delta)^2 = \text{id} \text{ (involution condition).}$$

**7.2 The Hopf algebra $\mathcal{H}_n$**

Let the manifold $M^n$ be affine flat (the $\mathbb{R}^n$ or the disjoint union of $\mathbb{R}^n$). The frame bundle is then trivial with $FM \simeq M \times \text{GL}_n(\mathbb{R})$. In local coordinates $(x^\mu)$ for $x \in U \subset M$, we can view the frame coordinates $x^\mu, y^\mu_j$ as a 1-jet of a map $\phi: \mathbb{R}^n \to \mathbb{R}^n$

$$\phi(t) = x + yt, \ x, t \in \mathbb{R}^n, \ y \in \text{GL}_n(\mathbb{R}),$$

where $(yt)^\mu = \sum_i y^\mu_i t^i$ for $t = (t^i) \in \mathbb{R}^n$.

We endow it with the trivial connection, given by the matrix-valued 1-form $\omega = (\omega^i_j)$, where

$$\omega^i_j := \sum_{\mu} (y^{-1})^i_{\mu} dy^\mu_j = (y^{-1}dy)^i_j$$

The corresponding basic horizontal fields on $FM$ are

$$X_k = \sum_{\mu} y^\mu_k \partial_{\mu}, \ k = 1, \ldots, n, \ \partial_{\mu} = \frac{\partial}{\partial x^\mu}.$$  

Denote by $\theta^k$ be the canonical form of the frame bundle

$$\theta^k := \sum_{\mu} (y^{-1})^k_{\mu} dx^\mu = (y^{-1}dx)^k, \ k = 1, \ldots, n.$$
Then let
\[ Y^j_i = \sum_{\mu} y^\mu_j \partial^\mu_i, \quad i, j = 1, \ldots, n, \quad \partial^\mu_i := \frac{\partial}{\partial y^\mu_j} \]
be the fundamental vertical vector fields associated to the standard basis of \( \mathfrak{gl}_n(\mathbb{R}) \) and generating the canonical right action of \( \text{GL}_n(\mathbb{R}) \) on \( FM \). At each point of \( FM \), \( \{X_k, Y^j_i\} \) and \( \{\theta^k, \omega^j_i\} \) form bases of the tangent and cotangent space, dual to each other
\[
\langle \omega^j_i, Y^l_k \rangle = \delta^j_i \delta^l_k, \quad \langle \omega^j_i, X_k \rangle = 0, \quad \langle \theta^k, Y^l_j \rangle = 0, \quad \langle \theta^k, X_j \rangle = \delta^k_j.
\]

The group of diffeomorphism \( \text{Diff}_M = \text{Diff}_{\mathbb{R}^n} \) acts on \( FM \) by the natural lift of the tautological action to the frame level
\[
\tilde{\varphi}(x, y) := (\varphi(x), \varphi'(x)y)
\]
where \( \varphi'(x) \) is Jacobi matrix \( \varphi'(x)^j_i = \frac{\partial \varphi^j_i}{\partial x^k} \).

Viewing \( \text{Diff}_M \) as a discrete group we form the crossed product algebra
\[
\mathfrak{A}_M := C^\infty_c(FM) \rtimes \text{Diff}_M
\]
As a vector space, it is spanned by monomials of the form \( f u^*_\varphi \), where \( f \in C^\infty(FM) \) and \( u^*_\varphi \) stands for \( \varphi^{-1} \). The product is given by
\[
f_1 u^*_\varphi_1 \cdot f_2 u^*_\varphi_2 = f_1(f_2 \circ \tilde{\varphi}_1)u^*_\varphi_2 \varphi_1.
\]
Since the right action of \( \text{GL}_n(\mathbb{R}) \) on \( FM \) commutes with the action of \( \text{Diff}_M \), at the Lie algebra level one has
\[
u_\varphi Y^j_i u^*_\varphi = Y^j_i.
\]
This allows to promote the vertical vector fields to derivations of \( \mathfrak{A}_M \). Indeed, setting
\[
Y^j_i(f u^*_\varphi) = Y^j_i(f)u^*_\varphi
\]
the extended operators satisfy the derivation rule
\[
Y^j_i(ab) = Y^j_i(a)b + aY^j_i(b), \quad a, b \in \mathfrak{A}_M.
\]
We shall also prolong the horizontal vector fields to linear transformations \( X_k \in \mathcal{L}(\mathfrak{A}_M) \) in similar fashion
\[
X_k(f u^*_\varphi) = X_k(f)u^*_\varphi.
\]
The resulting operators are no longer \( \text{Diff}_M \)-invariant. They satisfy
\[
u_\varphi X_k u^*_\varphi = X_k - \gamma^j_{ik}(\varphi^{-1})Y^j_i,
\]
where \( \varphi \mapsto \gamma^j_{ik}(\varphi) \) is a group 1-cocycle on \( \text{Diff}_M \) with values in \( C^\infty(FM) \). Specifically
\[
\gamma^j_{ik}(\varphi)(x, y) = \sum_{\mu} (y^{-1} \cdots \varphi'(x)^{-1} \cdot \partial^\mu \cdot y)^j_i \partial^\mu_k
\]
The above expression comes from the pull-back formula for the connection
\[
\tilde{\varphi}^*_\omega^j_i = \omega^j_i + \gamma^j_{ik}(\varphi)\theta^k.
\]
Now one uses the fact that \( \{ \theta^k, (\tilde{\omega}^{-1})^* (\omega^j) \} \) is the dual basis to \( \{ u^*_\varphi X_k u^*_\varphi, Y^j \varphi \} \).

As a consequence, the operators \( X_k \in \mathcal{L}(\mathfrak{A}_M) \) are no longer derivations of \( \mathfrak{A}_M \), but satisfy a non-symmetric Leibniz rule

\[
X_k(a, b) = X_k(a) b + a X_k(b) + \delta^j_{jk}(a) Y^j(b), \quad a, b \in \mathfrak{A}_M,
\]

where the linear operators \( \delta^j_{jk} \in \mathcal{L}(\mathfrak{A}_M) \) are defined by

\[
\delta^j_{jk}(f u^*_\varphi) = \gamma^i_{jk} f u^*_\varphi.
\]

These are derivations, i.e.

\[
\delta^j_{jk}(ab) = \delta^j_{jk}(a)b + a \delta^j_{jk}(b).
\]

The operators \( \{ X_k, Y^j \} \) satisfy the commutation relations of the group of affine transformations of \( \mathbb{R}^n \)

\[
[Y^j, Y^l] = \delta^l_j Y^l - \delta^l_j Y^l,
\]

\[
[Y^j, X_k] = \delta^j_k X_k,
\]

\[
[X_k, X_l] = 0.
\]

The successive commutators of the operators \( \delta^j_{jk} \) with the \( X_i \)'s yield new generations of

\[
\delta^j_{jk[l_1 \ldots l_r]} := [X_{l_r}, \ldots, [X_{l_1}, \delta^j_{jk}] \ldots],
\]

which involve multiplication by higher order jets of diffeomorphisms

\[
\delta^j_{jk[l_1 \ldots l_r]}(f u^*_\varphi) = \gamma^i_{jk[l_1 \ldots l_r]} f u^*_\varphi,
\]

where

\[
\delta^j_{jk[l_1 \ldots l_r]} := X_{l_r} \ldots X_{l_1} \left( \gamma^i_{jk} \right).
\]

They commute among themselves

\[
[\delta^j_{jk[l_1 \ldots l_r]}, \delta^{j'}_{j'k'[l'_1 \ldots l'_r]}] = 0.
\]

It can be checked that the order of \( \{ j, k \} \) and \( \{ l_1, \ldots, l_r \} \) does not matter - in any case we get the same operator.

The commutators between \( Y^\mu \)'s and \( \delta^j_{jk} \)'s can be obtained from explicit expression of the cocycle \( \gamma \), by computing its derivatives in the direction of the vertical vector fields. One obtains

\[
[Y^\mu, \delta^j_{jk}] = \delta^j_{j'k} \delta^\mu_{j'} + \delta^\mu_{j'} \delta^j_{jk} - \delta^j_j \delta^\mu_{k'}
\]

By induction

\[
[Y^\lambda_{\mu}, \delta^j_{jk[l_j \ldots l_r]}] = \sum_s \delta^\lambda_{j_s} \delta^j_{j_1 j_2 \ldots j_{s-1} j_s+1 \ldots j_r} - \delta^\mu_{j_1 j_2 j_3 \ldots j_r}.
\]

**Definition 7.4.** Let \( \mathcal{H}_n \) be the universal enveloping algebra of the Lie algebra \( \mathfrak{h}_n \) with basis

\[
\{ X_\lambda, Y^\mu_{\nu}, \delta^j_{jk[l_1 \ldots l_r]} \mid 1 \leq \lambda, \mu, \nu, i \leq n, 1 \leq j \leq k \leq n, 1 \leq l_1 \leq \ldots \leq l_r \leq n \}
\]

and the following presentation

\[
[X_k, X_l] = 0,
\]

\[
[Y^j, Y^l] = \delta^j_k Y^l - \delta^l_k Y^l,
\]

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\[ [Y_{ij}, X_k] = \delta_k^j X_i, \]
\[ [X_{ij}, \delta_{jk}^{i|1|...|r}] = \delta_{jk}^{i|1|...|r}, \]
\[ [Y_{ij}, \delta_{jk}^{i|l_1|...|l_r}] = \sum_{s=0}^{r} \delta_{jk}^{i|l_1|...|l_s} - \delta_{jk}^{i|l_1|...|l_r}, \]
\[ [\delta_{jk}^{i|1|...|l_r}, \delta_{jk'}^{i'|l_1'|...|l_r'}] = 0. \]

We shall endow \( H_n := U(h_n) \) with a canonical Hopf structure, which is noncommutative, and therefore different from the standard structure of a universal enveloping algebra.

**Proposition 7.5.**

1. The formulae

\[ \Delta X_k = X_k \otimes 1 + 1 \otimes X_k + \delta_{jk}^i Y_i^j, \]
\[ \Delta Y_i^j = Y_i^j \otimes 1 + 1 \otimes Y_i^j, \]
\[ \Delta \delta_{jk}^i = \delta_{jk}^i \otimes 1 + 1 \otimes \delta_{jk}^i, \]

uniquely determine a coproduct \( \Delta : H_n \to H_n \otimes H_n \), which makes \( H_n \) a bialgebra with respect to the product \( m : H_n \otimes H_n \to H_n \) and the counit \( \varepsilon : H_n \to \mathbb{C} \) inherited from \( U(h_n) \).

2. The formulae

\[ S(X_k) = -X_k + \delta_{jk}^i Y_i^j, \]
\[ S(Y_i^j) = -Y_i^j, \]
\[ S(\delta_{jk}^i) = -\delta_{jk}^i, \]

uniquely determine an anti-homomorphism \( S : H_n \to H_n \), which provides the antipode that turns \( H_n \) into a Hopf algebra.

The notation is justified while one proves that the subalgebra of \( \mathcal{L}(A_M) \) generated by the linear operators \( \{X_k, Y_i^j, \delta_{jk}^i \mid i, j, k = 1, \ldots, n \} \) is isomorphic to the algebra \( H_n \). The action of \( H_n \) turns \( A_n \) into a left \( H_n \)-module algebra. Moreover to any element \( h^1 \otimes \ldots \otimes h^p \in H_n^p \) we can associate a multilinear differential operator \( T \) acting on \( A_M \) as follows

\[ T(h^1 \otimes \ldots \otimes h^p)(a^1, \ldots, a^p) = h^1(a_1) \ldots h^p(a_p). \]

The linearization \( T : TH_n^p \to \mathcal{L}(A_M^\otimes^p, A_M) \) of this assignment is injective for each \( p \in \mathbb{N} \).